

## Solving Dual Fully Fuzzy Linear System with Octagonal Fuzzy Numbers

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### ABSTRACT

In this paper we proposed to solve the fully fuzzy linear system of the form  $\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$  where  $\tilde{A}_1$  and  $\tilde{A}_2$  are  $m \times n$  fuzzy matrices consisting of positive fuzzy numbers, the unknown vector  $\tilde{x}$  is a vector consisting of  $n$  positive fuzzy numbers and the constant  $\tilde{b}$  are vectors consisting of  $m$  positive fuzzy numbers, using Matrix Inversion method, Cramer's rule, Successive elimination methods such as Gaussian Elimination method and Gauss Jordan, Iteration methods such as Gauss Seidel and Gauss Jacobi methods, LU decomposition method, Cholesky decomposition method, QR decomposition method, Schur complement method, Linear programming approach, these methods gives the non negative solution of fully fuzzy linear system. In this paper we considered the fuzzy numbers are octagonal fuzzy numbers. Also we introduced some definitions for octagonal numbers.

**Key words:** Fully Fuzzy Linear system, Dual fully fuzzy linear system, Fuzzy matrix, Octagonal fuzzy numbers, Matrix inversion method, Cramer's rule, Gauss elimination, Gauss Jordan, Gauss Seidel, Gauss Jacobi, LU decomposition,  $LL^T$  decomposition, QR decomposition, Schur complement method, Linear programming approach.

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### 1. INTRODUCTION

System of simultaneous linear equations plays a vital role in mathematics, Operations Research, Statistics, Physics, Engineering and Social Sciences etc. In many applications at least some of the system's parameters and measurements are represented by fuzzy numbers rather than crisp numbers. Therefore it is imperative to develop mathematical models and numerical procedures to solve such a fuzzy linear system. The general model of a fuzzy linear system whose coefficient matrix is crisp and the right hand side column is an arbitrary fuzzy vector. In the fully fuzzy linear system all the parameters are considered to be fuzzy numbers. In this paper we considered square and non square dual fully fuzzy linear system of the form  $\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$  with octagonal fuzzy numbers which are non negative. M. Fridamn et al. [1] introduced a general model for solving a fuzzy  $n \times n$  linear system whose coefficient matrix is crisp and the right hand side column is a fuzzy vector of positive fuzzy numbers. T. Allahyiranloo [2] proposed solution of a fuzzy linear system by iterative methods such as Jacobi and Gauss Seidel methods. M. Dehghan et al. [3] are solved  $n \times n$  fully fuzzy linear system using direct method, Cramer's

rule, Gauss Elimination, Doolittle & Crout factorization methods and Linear programming approach. S.H. Naseri. et al. [4] proposed a method for solving fully fuzzy linear systems by certain decomposition (LU) of the coefficient matrix with triangular fuzzy numbers. M. Mosleh et al. [5] introduced ST decomposition for 2x2 fully fuzzy linear systems with triangular fuzzy numbers. S.H. Naseri et al.[7] have solved  $n \times n$  fully fuzzy linear system using Cholesky method. S.H. Nasserli et al. [6] proposed Greville's method to find the positive solution of fully fuzzy linear system. Amit Kumar et al. [8] discussed consistency of the fully fuzzy linear system and the nature of solutions. Amit kumar et al. [9] are bring in a new method for finding the non negative solution of fully fuzzy linear system without any restriction on the coefficient matrix. Amit kumar et al. [10] are introduced a new method for finding the non negative solution of the  $m \times n$  fully fuzzy linear system without any restriction on the coefficient matrix using Linear programming problem method.

### 1.1 The Structure of this paper is organized as follows

In Section 2, we present some basic concepts of fuzzy set theory and define a fully fuzzy linear system of equations. In Section 3, we have given the general model of dual fully fuzzy linear system. In Section 4, we extended classical methods such as Matrix Inversion method, Cramer's rule, Successive Elimination methods, Iterative methods, Decomposition methods, Schur complement method, linear programming approach for solving a fully fuzzy linear system In section 5 we illustrated numerical examples In section 6 gives Conclusion and References

## 2. PRELIMINARIES

**Definition 2.1** [11] An octagonal fuzzy number denoted by  $\tilde{A}_w$  is defined to be the ordered quadruple  $\tilde{A}_w = (l_1(r), s_1(t), s_2(t), l_2(r))$ , for  $r \in [0, k]$ , and  $t \in [k, w]$  where (i)  $l_1(r)$  is a bounded left continuous non decreasing function over  $[0, w_1]$ ,  $[0 \leq w_1 \leq k]$   
(ii)  $s_1(r)$  is a bounded left continuous non decreasing function over  $[k, w_2]$ ,  $[k \leq w_2 \leq w]$   
(iii)  $s_2(r)$  is a bounded left continuous non decreasing function over  $[k, w_2]$ ,  $[k \leq w_2 \leq w]$   
(iv)  $l_2(r)$  is a bounded left continuous non decreasing function over  $[0, w_1]$ ,  $[0 \leq w_1 \leq k]$

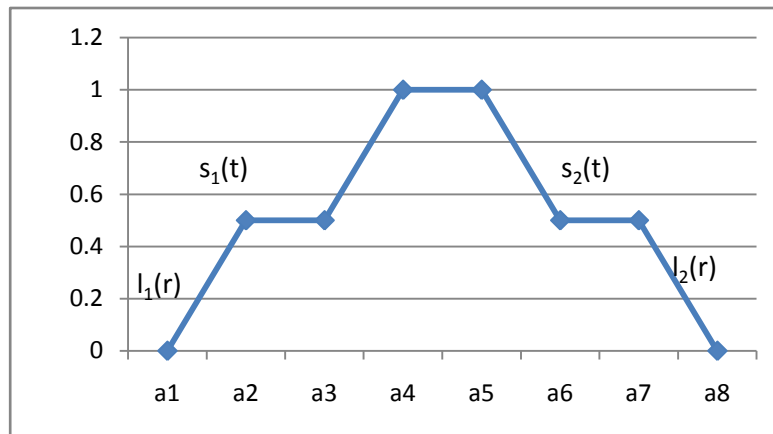
**Remark 2.2** [11] If  $w = 1$  then the above-defined number is called a normal octagonal fuzzy number.

**Definition 2.3** [11] A fuzzy number  $\tilde{A}$  is a normal octagonal fuzzy number denoted by  $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$  where  $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$  are real numbers and its membership function  $\mu_{\tilde{A}}(x)$  is given below

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{for } x < a_1 \\ k \left( \frac{x - a_1}{a_2 - a_1} \right), & \text{for } a_1 \leq x \leq a_2 \\ k, & \text{for } a_2 \leq x \leq a_3 \\ k + (1 - k) \left( \frac{x - a_3}{a_4 - a_3} \right), & \text{for } a_3 \leq x \leq a_4 \\ 1, & \text{for } a_4 \leq x \leq a_5 \\ k + (1 - k) \left( \frac{a_6 - x}{a_6 - a_5} \right), & \text{for } a_5 \leq x \leq a_6 \\ k, & \text{for } a_6 \leq x \leq a_7 \\ k \left( \frac{a_8 - x}{a_8 - a_7} \right), & \text{for } a_7 \leq x \leq a_8 \\ 0, & \text{for } x \geq a_8 \end{cases}$$

Where  $0 < k < 1$ .

Graphical representation of a normal octagonal fuzzy number for  $k = 0.5$  is



**Remark 2.4** [11] If  $k = 0$  then the octagonal fuzzy number reduces to the trapezoidal number  $(a_3, a_4, a_5, a_6)$  and if  $k = 1$ , it reduces to the  $(a_1, a_4, a_5, a_8)$ .

**Remark 2.5** [11] According to the above mentioned definition, octagonal fuzzy number  $\tilde{A}_w$  is the ordered quadruple  $(l_1(r), s_1(t), s_2(t), l_2(r))$  for  $r \in [0, k]$ , and  $t \in [k, w]$  where  $l_1(r) = k \left( \frac{r - a_1}{a_2 - a_1} \right)$ ,  $s_1(t) = k + (1 - k) \left( \frac{t - a_3}{a_4 - a_3} \right)$ ,  $s_2(t) = k + (1 - k) \left( \frac{a_6 - t}{a_6 - a_5} \right)$ , and  $l_2(r) = k \left( \frac{a_8 - r}{a_8 - a_7} \right)$ .

**Definition 2.6** [11] Membership functions  $\mu_{\tilde{A}_w}(x)$  are continuous functions.

**Remark 2.7** [11] Here  $\tilde{A}_w$  represents a fuzzy number in which “w” is the maximum membership value that a fuzzy number takes on. Whenever a normal fuzzy number is meant, the fuzzy number is shown  $\tilde{A}$ , for convenience.

**Remark 2.8** [11] If  $\tilde{A}_w$  be an octagonal fuzzy number, then the  $\alpha$ -cut of  $\tilde{A}_w$  is

$$\begin{aligned} [\tilde{A}_w]_\alpha &= \{x \mid \tilde{A}_w(x) \geq \alpha\} \\ &= \begin{cases} [l_1(\alpha), l_2(\alpha)] & \text{for } \alpha \in [0, k] \\ [s_1(\alpha), s_2(\alpha)] & \text{for } \alpha \in [k, w] \end{cases} \end{aligned}$$

**Remark 2.9** [11] The octagonal fuzzy number is convex as their  $\alpha$ -cuts are convex sets in the classical sense.

**Remark 2.10** [11] The collection of all octagonal fuzzy real numbers from  $\mathcal{R}$  to  $I$  is denoted as  $\mathcal{R}_w(I)$  and if  $w = 1$ , then the collection of normal octagonal fuzzy numbers is denoted by  $\mathcal{R}(I)$ .

**Definition 2.11** A fuzzy number  $\tilde{A}$  is called positive (negative), denoted by  $\tilde{A} > 0$  ( $\tilde{A} < 0$ ), if its membership function  $\mu_{\tilde{A}}(x)$  satisfies  $\mu_{\tilde{A}}(x) = 0, \forall x \leq 0$  ( $\forall x \geq 0$ ).

**Definition 2.12** Two octagonal fuzzy numbers  $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$  &  $\tilde{B} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$  are said to be equal if and only if  $a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4, a_5 = b_5, a_6 = b_6, a_7 = b_7, a_8 = b_8$ .

**Definition 2.13** An octagonal fuzzy number  $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$  is said to be zero octagonal fuzzy number if and only if  $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = 0, a_7 = 0, a_8 = 0$ .

**Definition 2.14** Let  $\tilde{A} = (\tilde{a}_{ij})$  and  $\tilde{B} = (\tilde{b}_{ij})$  be two  $m \times n$  and  $n \times p$  fuzzy matrices. We define  $\tilde{A} \otimes \tilde{B} = \tilde{C} = (\tilde{c}_{ij})$  Which is the  $m \times p$  matrix where  $\tilde{c}_{ij} = \sum_{k=1,2,\dots,n}^{\oplus} \tilde{a}_{ik} \otimes \tilde{b}_{kj}$

**Definition 2.15** Arithmetic operations on octagonal fuzzy numbers

Let  $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$  and  $\tilde{B} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$  be two octagonal fuzzy numbers then

- (i)  $\tilde{A} \oplus \tilde{B} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \oplus (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$   
 $= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5, a_6 + b_6, a_7 + b_7, a_8 + b_8)$
- (ii)  $\tilde{A} \geq 0$  and  $\tilde{B} \geq 0$  then  $\tilde{A} \otimes \tilde{B} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \otimes (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$   
 $= (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4, a_5 b_5, a_6 b_6, a_7 b_7, a_8 b_8)$
- (iii)  $\tilde{A} \ominus \tilde{B} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8) \ominus (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$   
 $= (a_1 - b_8, a_2 - b_7, a_3 - b_6, a_4 - b_5, a_5 - b_4, a_6 - b_3, a_7 - b_2, a_8 - b_1)$

**Definition 2.16** A matrix  $\tilde{A} = (\tilde{a}_{ij})$  is called a fuzzy matrix, if each element of  $\tilde{A}$  is a fuzzy number. A fuzzy matrix  $\tilde{A}$  will be positive and denoted by  $\tilde{A} > 0$ , if each element of  $\tilde{A}$  be positive. We may represent  $n \times n$  fuzzy matrix  $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ , such that  $\tilde{a}_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}, g_{ij}, h_{ij})$ , with the new notation  $\tilde{A} = (A, B, C, D, E, F, G, H)$ , where  $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij}), E = (e_{ij}), F = (f_{ij}), G = (g_{ij}), H = (h_{ij})$  are eight  $n \times n$  crisp matrices.

**Definition 2.17** A square fuzzy matrix  $\tilde{A} = (\tilde{a}_{ij})$  will be an upper triangular fuzzy matrix, if  $\tilde{a}_{ij} = \tilde{0} = (0,0,0,0,0,0,0,0) \forall i > j$ , and a square fuzzy matrix  $\tilde{A} = (\tilde{a}_{ij})$  will be a lower triangular fuzzy matrix, if  $\tilde{a}_{ij} = \tilde{0} = (0,0,0,0,0,0,0,0) \forall i < j$ .

**Definition 2.18** Consider the  $n \times n$  fuzzy linear system of equations

$$(\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1$$

$$(\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2$$

$$\dots$$

$$\dots$$

$$\dots$$

$$(\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) = \tilde{b}_n$$

The matrix form of the above equations is  $\tilde{A} \otimes \tilde{x} = \tilde{b}$  **(2.18.1)**

Where the coefficient matrix  $\tilde{A} = (\tilde{a}_{ij})$ ,  $1 \leq i, j \leq n$  is a  $n \times n$  fuzzy matrix and  $\tilde{x}_j, \tilde{b}_j \in F(R)$ . This system is called a fully fuzzy linear system.

In this paper  $\tilde{A} \otimes \tilde{x} = \tilde{b}$  where  $\tilde{A} = (A, B, C, D, E, F, G, H) \geq 0$ ,  
 $\tilde{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \geq 0$  and  $\tilde{b} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8) \geq 0$

We have

$$(A, B, C, D, E, F, G, H) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$$

Using 2.15 (ii) we have

$$(Ax_1, Bx_2, Cx_3, Dx_4, Ex_5, Fx_6, Gx_7, Hx_8) = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$$

Using 2.12 we have

$$\begin{aligned} Ax_1 &= b_1, & Bx_2 &= b_2, & Cx_3 &= b_3, & Dx_4 &= b_4, \\ Ex_5 &= b_5, & Fx_6 &= b_6, & Gx_7 &= b_7, & Hx_8 &= b_8 \end{aligned} \quad \mathbf{(2.18.2)}$$

By assuming that  $A, B, C, D, E, F, G, H$  are non-singular matrices we have

$$\begin{aligned} x_1 &= A^{-1}b_1, & x_2 &= B^{-1}b_2, & x_3 &= C^{-1}b_3, & x_4 &= D^{-1}b_4, \\ x_5 &= E^{-1}b_5, & x_6 &= F^{-1}b_6, & x_7 &= G^{-1}b_7, & x_8 &= H^{-1}b_8 \end{aligned} \quad \mathbf{(2.18.3)}$$

**Definition 2.19** A square matrix  $A$  is symmetric if and only if  $A = A^T$ .

**Definition 2.20** A real symmetric  $n \times n$  matrix  $A$  is said to be positive definite if  $X^T A X > 0$ , for all nonzero  $X$  in  $R^n$ .

**Definition 2.21** If  $A$  is positive definite matrix, then the leading principal sub matrices  $A_1, A_2, \dots, A_n$  of  $A$  are all positive definite.

**Definition 2.22** Let  $A$  be a positive definite matrix then there exists a unique lower triangular matrix  $L$  with positive diagonal entries such that  $A = LL^T$ .

**Definition 2.23** If  $A$  is an  $m \times n$  matrix with full column rank, then  $A$  ( $m \geq n$ ) can be factored as  $A = QR$  where  $Q$  is an  $m \times n$  matrix whose column vectors form an orthonormal basis for the column space  $A$  and  $R$  is a  $n \times n$  invertible upper triangular matrix.

### 3. DUAL FULLY FUZZY LINEAR SYSTEM

There is no inverse with respect to addition element for an arbitrary fuzzy number  $\tilde{u} \in E'$ .

That is there exists no element  $\tilde{v} \in E'$  such that  $\tilde{u} \oplus \tilde{v} = 0$

For all non crisp fuzzy numbers  $\tilde{u} \in E'$  we have  $\tilde{u} \oplus (-\tilde{u}) \neq 0$ .

Therefore the fully fuzzy linear system  $\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$  can't equivalently replaced by the fully fuzzy linear system  $(\tilde{A}_1 - \tilde{A}_2) \otimes \tilde{x} = \tilde{b}$

Hence the fully fuzzy linear system is of the form  $\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$  is called dual fully fuzzy linear system.

In this paper we are going to find a solution of dual fully fuzzy linear system

$$\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b} \quad (3.1)$$

Where  $\tilde{A}_1 = (A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1)$ ,  $\tilde{A}_2 = (A_2, B_2, C_2, D_2, E_2, F_2, G_2, H_2)$ ,

$\tilde{b} = (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$  and  $\tilde{x} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$

$\tilde{A}_1, \tilde{A}_2, \tilde{b}$  and  $\tilde{x} \geq 0$

$$(A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (A_2, B_2, C_2, D_2, E_2, F_2, G_2, H_2) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$$

Using 2.15 (ii) we get

$$(A_1x_1, B_1x_2, C_1x_3, D_1x_4, E_1x_5, F_1x_6, G_1x_7, H_1x_8) = (A_2x_1, B_2x_2, C_2x_3, D_2x_4, E_2x_5, F_2x_6, G_2x_7, H_2x_8) \oplus (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$$

Using 2.15 (i) we get

$$(A_1x_1, B_1x_2, C_1x_3, D_1x_4, E_1x_5, F_1x_6, G_1x_7, H_1x_8) = (A_2x_1 + b_1, B_2x_2 + b_2, C_2x_3 + b_3, D_2x_4 + b_4, E_2x_5 + b_5, F_2x_6 + b_6, G_2x_7 + b_7, H_2x_8 + b_8)$$

Using 2.12 we have

$$\begin{aligned} A_1x_1 &= A_2x_1 + b_1 \\ \Rightarrow (A_1 - A_2)x_1 &= b_1 \\ B_1x_2 &= B_2x_2 + b_2 \\ \Rightarrow (B_1 - B_2)x_2 &= b_2 \end{aligned}$$

$$C_1 x_3 = C_2 x_3 + b_3 \\ \Rightarrow (C_1 - C_2) x_3 = b_3$$

$$D_1 x_4 = D_2 x_4 + b_4 \\ \Rightarrow (D_1 - D_2) x_4 = b_4$$

$$E_1 x_5 = E_2 x_5 + b_5 \\ \Rightarrow (E_1 - E_2) x_5 = b_5 \\ F_1 x_6 = F_2 x_6 + b_6 \\ \Rightarrow (F_1 - F_2) x_6 = b_6$$

$$G_1 x_7 = G_2 x_7 + b_7 \\ \Rightarrow (G_1 - G_2) x_7 = b_7$$

$$H_1 x_8 = H_2 x_8 + b_8 \\ \Rightarrow (H_1 - H_2) x_8 = b_8$$

Let us take  $(A_1 - A_2) = A$ ,  $(B_1 - B_2) = B$ ,  $(C_1 - C_2) = C$ ,  $(D_1 - D_2) = D$ ,  $(E_1 - E_2) = E$ ,  $(F_1 - F_2) = F$ ,  $(G_1 - G_2) = G$ ,  $(H_1 - H_2) = H$ , the above equations becomes

$$\begin{aligned} Ax_1 &= b_1, & Bx_2 &= b_2, & Cx_3 &= b_3, & Dx_4 &= b_4, \\ Ex_5 &= b_5, & Fx_6 &= b_6, & Gx_7 &= b_7, & Hx_8 &= b_8 \end{aligned} \quad (3.2)$$

By assuming that  $A, B, C, D, E, F, G, H$  are non-singular matrices we have

$$\begin{aligned} x_1 &= A^{-1}b_1, & x_2 &= B^{-1}b_2, & x_3 &= C^{-1}b_3, & x_4 &= D^{-1}b_4, \\ x_5 &= E^{-1}b_5, & x_6 &= F^{-1}b_6, & x_7 &= G^{-1}b_7, & x_8 &= H^{-1}b_8 \end{aligned} \quad (3.3)$$

## 4. SOLVING DUAL FULLY FUZZY LINEAR SYSTEM

### 4.1. Matrix Inversion Method

For solving dual fully fuzzy linear system (3.1) with this method, Consider (3.2), Thus we may have

$$\begin{aligned} x_1 &= A^{-1}b_1, & x_2 &= B^{-1}b_2, & x_3 &= C^{-1}b_3, & x_4 &= D^{-1}b_4, \\ x_5 &= E^{-1}b_5, & x_6 &= F^{-1}b_6, & x_7 &= G^{-1}b_7, & x_8 &= H^{-1}b_8 \end{aligned}$$

Where  $A, B, C, D, E, F, G, H$  are non-singular matrices

### 4.2 Cramer's rule

For solving dual fully fuzzy linear system (3.1) with this method, consider (3.2). Thus we may write

$$x_{1_i} = \frac{\det(A^{(i)})}{\det(A)}, \quad i = 1, 2, \dots, n, \quad \det(A) \neq 0$$

Where  $A^{(i)}$  denotes the matrices obtained from  $A$  by replacing its  $i$ th column by  $b_i$ .

$$x_{2_i} = \frac{\det(B^{(i)})}{\det(B)}, i = 1, 2, \dots, n, \det(B) \neq 0$$

Where  $B^{(i)}$  denotes the matrices obtained from B by replacing its  $i$ th column by  $b_2$ .

$$x_{3_i} = \frac{\det(C^{(i)})}{\det(C)}, i = 1, 2, \dots, n, \det(C) \neq 0$$

Where  $C^{(i)}$  denotes the matrices obtained from C by replacing its  $i$ th column by  $b_3$ .

$$x_{4_i} = \frac{\det(D^{(i)})}{\det(D)}, i = 1, 2, \dots, n, \det(D) \neq 0$$

Where  $D^{(i)}$  denotes the matrices obtained from D by replacing its  $i$ th column by  $b_4$ .

$$x_{5_i} = \frac{\det(E^{(i)})}{\det(E)}, i = 1, 2, \dots, n, \det(E) \neq 0$$

Where  $E^{(i)}$  denotes the matrices obtained from E by replacing its  $i$ th column by  $b_5$ .

$$x_{6_i} = \frac{\det(F^{(i)})}{\det(F)}, i = 1, 2, \dots, n, \det(F) \neq 0$$

Where  $F^{(i)}$  denotes the matrices obtained from F by replacing its  $i$ th column by  $b_6$ .

$$x_{7_i} = \frac{\det(G^{(i)})}{\det(G)}, i = 1, 2, \dots, n, \det(G) \neq 0$$

Where  $G^{(i)}$  denotes the matrices obtained from G by replacing its  $i$ th column by  $b_7$ .

$$x_{8_i} = \frac{\det(H^{(i)})}{\det(H)}, i = 1, 2, \dots, n, \det(H) \neq 0$$

Where  $H^{(i)}$  denotes the matrices obtained from H by replacing its  $i$ th column by  $b_8$ .

### 4.3. Successive Eliminations methods

#### 4.3.1. Gaussian-Elimination method

A Well-Known procedure for solving the problem  $Ax = b$  is the classical elimination scheme known as Gaussian elimination, Which is now extended for solving fully fuzzy linear system in (3.1). The Basic idea is to reduce the system to an equivalent upper triangular fuzzy system so that the reduced form can be solved easily using back substitution process. Consider the equation (3.2) and apply the above said procedure.

#### 4.3.2. Gaussian-Jordan method

This method is modified version of Gauss-elimination method. Here the elimination of unknowns are not only performed in equations below but in equations above also, ultimately to

get a diagonal coefficient matrix, Which is now extended for solving fully fuzzy linear system in (3.1). Consider the equation (3.2) and apply the above said procedure.

#### 4.4. Iterative methods

A linear system of equations  $Ax = b$  can be written in the equivalent form  $x = Bx + c$ . By taking some initial vector  $x^{(0)}$  and compute  $x^{(1)} = Bx^{(0)} + c$ .

In general iterative rule  $x^{(m+1)} = Bx^{(m)} + c$ .  $m=0,1,2,\dots$  generates the sequence of the vectors, which converges to the solution vector  $\xi$ , under some suitable conditions. Such methods are known as iterative methods.

**Result 4.4.1:** Let  $Ax = b$  be written as  $x = Bx + c$ , with some norm of  $B$ ,  $\|B\| < 1$ , then  $x = Bx + c$  has unique solution. Further, sequence  $\{x^{(m)}\}$  generated by  $x^{(m+1)} = Bx^{(m)} + c$ , starting with some initial  $x^{(0)}$  will converge to true solution vector  $\xi$ .

**Result 4.4.2:** Jacobi's method to solve  $Ax = b$  converges, if coefficient matrix  $A$  is strictly diagonally row dominant.

#### 4.4.3. Gauss-Jacobi method

In matrix form  $Ax = b \Rightarrow (L+D+U)x = b$ , where  $L$ ,  $D$ ,  $U$  are lower triangular, diagonal and upper triangular respectively.

$$Dx = -(L+U)x + b$$

$$x = -D^{-1}(L+U)x + D^{-1}b$$

Since  $D$  is a diagonal row dominant. Therefore

$$x^{(m+1)} = -D^{-1}(L+U)x^{(m)} + D^{-1}b$$

Algorithm of Jacobi method

$$x_i^{(m+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(m)} - \sum_{j=i+1}^n a_{ij}x_j^{(m)} \right)$$

$$m=0,1,2,\dots$$

$$i=1,2,3,\dots,n$$

Consider the equation (3.2) and apply the above algorithm.

#### 4.4.4. Gauss-Seidel method

Gauss Seidel is a modification of Jacobi's method while computing  $x_i^{(m+1)}$ ,  $(m+1)^{th}$  iteration vector, the values of  $x^{(m)}$ ,  $(m)^{th}$  iterated vectors are used on right side. In Gauss-Seidel, latest value of  $x_i$  are used to compute  $x_i^{(m+1)}$ . To compute  $x_i^{(m+1)}$ , latest values of

$$x_1^{(m+1)}, x_2^{(m+1)}, \dots, x_{i-1}^{(m+1)}$$

are available and  $x_{i+1}^{(m)}, x_{i+2}^{(m)}, \dots$  from the previous set. This can be expressed as

$$x_i^{(m+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(m+1)} - \sum_{j=i+1}^n a_{ij}x_j^{(m)} \right)$$

$$i=1,2,\dots,n$$

In matrix form  $Ax = b \Rightarrow (L+D+U)x = b$

$$x + D^{-1}Lx = -D^{-1}Ux + D^{-1}b$$

$$x^{(m+1)} + D^{-1}Lx^{(m+1)} = -D^{-1}Ux^{(m)} + D^{-1}b$$

$$x^{(m+1)} = -D^{-1}(Lx^{(m+1)} + Ux^{(m)}) + D^{-1}b$$

Consider the equation (3.2) and apply the above algorithm.

#### 4.5. Simplification of LU factorization

Let A be an  $n \times n$  matrix with all non-zero leading principal minors. Then A has a unique factorization.  $A = LU$ , Where L is a unit lower triangular unit matrix and U is an upper triangular unit matrix.

Let

$$(A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1) = (L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8) \otimes (U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8) \\ = (L_1 U_1, L_2 U_2, L_3 U_3, L_4 U_4, L_5 U_5, L_6 U_6, L_7 U_7, L_8 U_8) \quad (4.5.1)$$

Where matrices  $L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8$  are lower triangular crisp matrices,  $U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8$  are upper triangular crisp matrices.

Let

$$(A_2, B_2, C_2, D_2, E_2, F_2, G_2, H_2) = \\ (L_1^{\square}, L_2^{\square}, L_3^{\square}, L_4^{\square}, L_5^{\square}, L_6^{\square}, L_7^{\square}, L_8^{\square}) \otimes (U_1^{\square}, U_2^{\square}, U_3^{\square}, U_4^{\square}, U_5^{\square}, U_6^{\square}, U_7^{\square}, U_8^{\square}) \\ = (L_1^{\square} U_1^{\square}, L_2^{\square} U_2^{\square}, L_3^{\square} U_3^{\square}, L_4^{\square} U_4^{\square}, L_5^{\square} U_5^{\square}, L_6^{\square} U_6^{\square}, L_7^{\square} U_7^{\square}, L_8^{\square} U_8^{\square}) \quad (4.5.2)$$

Where matrices  $L_1^{\square}, L_2^{\square}, L_3^{\square}, L_4^{\square}, L_5^{\square}, L_6^{\square}, L_7^{\square}, L_8^{\square}$  are lower triangular crisp matrices,  $U_1^{\square}, U_2^{\square}, U_3^{\square}, U_4^{\square}, U_5^{\square}, U_6^{\square}, U_7^{\square}, U_8^{\square}$  are upper triangular crisp matrices.

For solving dual fully fuzzy linear system (3.1) with this method.

$$\text{Consider } \tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$$

$$(A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \\ (A_2, B_2, C_2, D_2, E_2, F_2, G_2, H_2) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$$

Where  $A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1, A_2, B_2, C_2, D_2, E_2, F_2, G_2, H_2$  are non singular matrices

Using (4.5.1) and (4.5.2) we have

$$(L_1 U_1, L_2 U_2, L_3 U_3, L_4 U_4, L_5 U_5, L_6 U_6, L_7 U_7, L_8 U_8) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \\ (L_1^{\square} U_1^{\square}, L_2^{\square} U_2^{\square}, L_3^{\square} U_3^{\square}, L_4^{\square} U_4^{\square}, L_5^{\square} U_5^{\square}, L_6^{\square} U_6^{\square}, L_7^{\square} U_7^{\square}, L_8^{\square} U_8^{\square}) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \\ \oplus (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$$

Using 2.15(ii) we have

$$(L_1 U_1 x_1, L_2 U_2 x_2, L_3 U_3 x_3, L_4 U_4 x_4, L_5 U_5 x_5, L_6 U_6 x_6, L_7 U_7 x_7, L_8 U_8 x_8) = \\ (L_1^{\square} U_1^{\square} x_1, L_2^{\square} U_2^{\square} x_2, L_3^{\square} U_3^{\square} x_3, L_4^{\square} U_4^{\square} x_4, L_5^{\square} U_5^{\square} x_5, L_6^{\square} U_6^{\square} x_6, L_7^{\square} U_7^{\square} x_7, L_8^{\square} U_8^{\square} x_8) \oplus (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$$

Using 2.15(i) we have

$$(L_1 U_1 x_1, L_2 U_2 x_2, L_3 U_3 x_3, L_4 U_4 x_4, L_5 U_5 x_5, L_6 U_6 x_6, L_7 U_7 x_7, L_8 U_8 x_8) = \\ (L_1^{\square} U_1^{\square} x_1 + b_1, L_2^{\square} U_2^{\square} x_2 + b_2, L_3^{\square} U_3^{\square} x_3 + b_3, L_4^{\square} U_4^{\square} x_4 + b_4, L_5^{\square} U_5^{\square} x_5 + b_5, L_6^{\square} U_6^{\square} x_6 + b_6, L_7^{\square} U_7^{\square} x_7 + b_7, L_8^{\square} U_8^{\square} x_8 + b_8)$$

Using 2.12 we have

$$L_1 U_1 x_1 = L_1^{\blacksquare} U_1^{\blacksquare} x_1 + b_1 \\ \Rightarrow x_1 = (L_1 U_1 - L_1^{\blacksquare} U_1^{\blacksquare})^{-1} b_1$$

$$L_2 U_2 x_2 = L_2^{\blacksquare} U_2^{\blacksquare} x_2 + b_2 \\ \Rightarrow x_2 = (L_2 U_2 - L_2^{\blacksquare} U_2^{\blacksquare})^{-1} b_2$$

$$L_3 U_3 x_3 = L_3^{\blacksquare} U_3^{\blacksquare} x_3 + b_3 \\ \Rightarrow x_3 = (L_3 U_3 - L_3^{\blacksquare} U_3^{\blacksquare})^{-1} b_3$$

$$L_4 U_4 x_4 = L_4^{\blacksquare} U_4^{\blacksquare} x_4 + b_4 \\ \Rightarrow x_4 = (L_4 U_4 - L_4^{\blacksquare} U_4^{\blacksquare})^{-1} b_4$$

$$L_5 U_5 x_5 = L_5^{\blacksquare} U_5^{\blacksquare} x_5 + b_5 \\ \Rightarrow x_5 = (L_5 U_5 - L_5^{\blacksquare} U_5^{\blacksquare})^{-1} b_5$$

$$L_6 U_6 x_6 = L_6^{\blacksquare} U_6^{\blacksquare} x_6 + b_6 \\ \Rightarrow x_6 = (L_6 U_6 - L_6^{\blacksquare} U_6^{\blacksquare})^{-1} b_6$$

$$L_7 U_7 x_7 = L_7^{\blacksquare} U_7^{\blacksquare} x_7 + b_7 \\ \Rightarrow x_7 = (L_7 U_7 - L_7^{\blacksquare} U_7^{\blacksquare})^{-1} b_7$$

$$L_8 U_8 x_8 = L_8^{\blacksquare} U_8^{\blacksquare} x_8 + b_8 \\ \Rightarrow x_8 = (L_8 U_8 - L_8^{\blacksquare} U_8^{\blacksquare})^{-1} b_1$$

#### 4.6. Simplification of $LL^T$ factorization

Let  $A$  be an  $n \times n$  non singular symmetric positive definite matrix with all non-zero leading principal minors. Then  $A$  has a unique factorization.  $A = LL^T$ , Where  $L$  is a unit lower triangular unit matrix.

$$\text{Let } (A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1) = (L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8) \otimes (L_1^T, L_2^T, L_3^T, L_4^T, L_5^T, L_6^T, L_7^T, L_8^T) \\ = (L_1 L_1^T, L_2 L_2^T, L_3 L_3^T, L_4 L_4^T, L_5 L_5^T, L_6 L_6^T, L_7 L_7^T, L_8 L_8^T) \quad (4.6.1)$$

Where matrices  $L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8$  are lower triangular crisp matrices

$$\text{Let } (A_2, B_2, C_2, D_2, E_2, F_2, G_2, H_2) = (L_1^{\blacksquare}, L_2^{\blacksquare}, L_3^{\blacksquare}, L_4^{\blacksquare}, L_5^{\blacksquare}, L_6^{\blacksquare}, L_7^{\blacksquare}, L_8^{\blacksquare}) \otimes ((L_1^{\blacksquare})^T, (L_2^{\blacksquare})^T, (L_3^{\blacksquare})^T, (L_4^{\blacksquare})^T, (L_5^{\blacksquare})^T, (L_6^{\blacksquare})^T, (L_7^{\blacksquare})^T, (L_8^{\blacksquare})^T) \\ = (L_1^{\blacksquare} (L_1^{\blacksquare})^T, L_2^{\blacksquare} (L_2^{\blacksquare})^T, L_3^{\blacksquare} (L_3^{\blacksquare})^T, L_4^{\blacksquare} (L_4^{\blacksquare})^T, L_5^{\blacksquare} (L_5^{\blacksquare})^T, L_6^{\blacksquare} (L_6^{\blacksquare})^T, L_7^{\blacksquare} (L_7^{\blacksquare})^T, L_8^{\blacksquare} (L_8^{\blacksquare})^T) \quad (4.6.2)$$

Where matrices  $L_1^{\blacksquare}, L_2^{\blacksquare}, L_3^{\blacksquare}, L_4^{\blacksquare}, L_5^{\blacksquare}, L_6^{\blacksquare}, L_7^{\blacksquare}, L_8^{\blacksquare}$  are lower triangular crisp matrices,

For solving dual fully fuzzy linear system (3.1) with this method.

$$\text{Consider } \tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$$

$$(A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \\ (A_2, B_2, C_2, D_2, E_2, F_2, G_2, H_2) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$$

Where  $A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1, A_2, B_2, C_2, D_2, E_2, F_2, G_2, H_2$  are non singular matrices

Using (4.5.1) and (4.5.2) we have

$$(L_1 L_1^T, L_2 L_2^T, L_3 L_3^T, L_4 L_4^T, L_5 L_5^T, L_6 L_6^T, L_7 L_7^T, L_8 L_8^T) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \\ (L_1^{\blacksquare} (L_1^{\blacksquare})^T, L_2^{\blacksquare} (L_2^{\blacksquare})^T, L_3^{\blacksquare} (L_3^{\blacksquare})^T, L_4^{\blacksquare} (L_4^{\blacksquare})^T, L_5^{\blacksquare} (L_5^{\blacksquare})^T, L_6^{\blacksquare} (L_6^{\blacksquare})^T, L_7^{\blacksquare} (L_7^{\blacksquare})^T, L_8^{\blacksquare} (L_8^{\blacksquare})^T) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \\ \oplus (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$$

Using 2.15(ii) we have

$$(L_1 L_1^T x_1, L_2 L_2^T x_2, L_3 L_3^T x_3, L_4 L_4^T x_4, L_5 L_5^T x_5, L_6 L_6^T x_6, L_7 L_7^T x_7, L_8 L_8^T x_8) = \\ (L_1^{\blacksquare} (L_1^{\blacksquare})^T x_1, L_2^{\blacksquare} (L_2^{\blacksquare})^T x_2, L_3^{\blacksquare} (L_3^{\blacksquare})^T x_3, L_4^{\blacksquare} (L_4^{\blacksquare})^T x_4, L_5^{\blacksquare} (L_5^{\blacksquare})^T x_5, L_6^{\blacksquare} (L_6^{\blacksquare})^T x_6, L_7^{\blacksquare} (L_7^{\blacksquare})^T x_7, L_8^{\blacksquare} (L_8^{\blacksquare})^T x_8) \\ \oplus (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$$

Using 2.15(i) we have

$$(L_1 L_1^T x_1, L_2 L_2^T x_2, L_3 L_3^T x_3, L_4 L_4^T x_4, L_5 L_5^T x_5, L_6 L_6^T x_6, L_7 L_7^T x_7, L_8 L_8^T x_8) = \\ (L_1^{\blacksquare} (L_1^{\blacksquare})^T x_1 + b_1, L_2^{\blacksquare} (L_2^{\blacksquare})^T x_2 + b_2, L_3^{\blacksquare} (L_3^{\blacksquare})^T x_3 + b_3, L_4^{\blacksquare} (L_4^{\blacksquare})^T x_4 + b_4, L_5^{\blacksquare} (L_5^{\blacksquare})^T x_5 + \\ b_5, L_6^{\blacksquare} (L_6^{\blacksquare})^T x_6 + b_6, L_7^{\blacksquare} (L_7^{\blacksquare})^T x_7 + b_7, L_8^{\blacksquare} (L_8^{\blacksquare})^T x_8 + b_8)$$

Using 2.12 we have

$$L_1 L_1^T x_1 = L_1^{\blacksquare} (L_1^{\blacksquare})^T x_1 + b_1 \\ \Rightarrow x_1 = (L_1 L_1^T - L_1^{\blacksquare} (L_1^{\blacksquare})^T)^{-1} b_1$$

$$L_2 L_2^T x_2 = L_2^{\blacksquare} (L_2^{\blacksquare})^T x_2 + b_2 \\ \Rightarrow x_2 = (L_2 L_2^T - L_2^{\blacksquare} (L_2^{\blacksquare})^T)^{-1} b_2$$

$$L_3 L_3^T x_3 = L_3^{\blacksquare} (L_3^{\blacksquare})^T x_3 + b_3 \\ \Rightarrow x_3 = (L_3 L_3^T - L_3^{\blacksquare} (L_3^{\blacksquare})^T)^{-1} b_3$$

$$L_4 L_4^T x_4 = L_4^{\blacksquare} (L_4^{\blacksquare})^T x_4 + b_4 \\ \Rightarrow x_4 = (L_4 L_4^T - L_4^{\blacksquare} (L_4^{\blacksquare})^T)^{-1} b_4$$

$$L_5 L_5^T x_5 = L_5^{\blacksquare} (L_5^{\blacksquare})^T x_5 + b_5 \\ \Rightarrow x_5 = (L_5 L_5^T - L_5^{\blacksquare} (L_5^{\blacksquare})^T)^{-1} b_5$$

$$L_6 L_6^T x_6 = L_6^{\blacksquare} (L_6^{\blacksquare})^T x_6 + b_6 \\ \Rightarrow x_6 = (L_6 L_6^T - L_6^{\blacksquare} (L_6^{\blacksquare})^T)^{-1} b_6$$

$$L_7 L_7^T x_7 = L_7^{\blacksquare} (L_7^{\blacksquare})^T x_7 + b_7 \\ \Rightarrow x_7 = (L_7 L_7^T - L_7^{\blacksquare} (L_7^{\blacksquare})^T)^{-1} b_7$$

$$L_8 L_8^T x_8 = L_8^{\blacksquare} (L_8^{\blacksquare})^T x_8 + b_8 \\ \Rightarrow x_8 = (L_8 L_8^T - L_8^{\blacksquare} (L_8^{\blacksquare})^T)^{-1} b_8$$

#### 4.7. Simplification of QR factorization

$$\text{Let } (A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1) = \\ (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8) \otimes (R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8) \\ = (Q_1 R_1, Q_2 R_2, Q_3 R_3, Q_4 R_4, Q_5 R_5, Q_6 R_6, Q_7 R_7, Q_8 R_8) \quad (4.7.1)$$

$$\text{Let } (A_2, B_2, C_2, D_2, E_2, F_2, G_2, H_2) = \\ (Q_1^{\blacksquare}, Q_2^{\blacksquare}, Q_3^{\blacksquare}, Q_4^{\blacksquare}, Q_5^{\blacksquare}, Q_6^{\blacksquare}, Q_7^{\blacksquare}, Q_8^{\blacksquare}) \otimes (R_1^{\blacksquare}, R_2^{\blacksquare}, R_3^{\blacksquare}, R_4^{\blacksquare}, R_5^{\blacksquare}, R_6^{\blacksquare}, R_7^{\blacksquare}, R_8^{\blacksquare}) \\ = (Q_1^{\blacksquare} R_1^{\blacksquare}, Q_2^{\blacksquare} R_2^{\blacksquare}, Q_3^{\blacksquare} R_3^{\blacksquare}, Q_4^{\blacksquare} R_4^{\blacksquare}, Q_5^{\blacksquare} R_5^{\blacksquare}, Q_6^{\blacksquare} R_6^{\blacksquare}, Q_7^{\blacksquare} R_7^{\blacksquare}, Q_8^{\blacksquare} R_8^{\blacksquare}) \quad (4.7.2)$$

For solving dual fully fuzzy linear system (3.1) with this method.

$$\text{Consider } \tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$$

$$(A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \\ (A_2, B_2, C_2, D_2, E_2, F_2, G_2, H_2) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$$

Where  $A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1, A_2, B_2, C_2, D_2, E_2, F_2, G_2, H_2$  are square matrices

Using (4.5.1) and (4.5.2) we have

$$(Q_1 R_1, Q_2 R_2, Q_3 R_3, Q_4 R_4, Q_5 R_5, Q_6 R_6, Q_7 R_7, Q_8 R_8) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \\ (Q_1^{\blacksquare} R_1^{\blacksquare}, Q_2^{\blacksquare} R_2^{\blacksquare}, Q_3^{\blacksquare} R_3^{\blacksquare}, Q_4^{\blacksquare} R_4^{\blacksquare}, Q_5^{\blacksquare} R_5^{\blacksquare}, Q_6^{\blacksquare} R_6^{\blacksquare}, Q_7^{\blacksquare} R_7^{\blacksquare}, Q_8^{\blacksquare} R_8^{\blacksquare}) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \\ \oplus (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$$

Using 2.15(ii) we have

$$(Q_1 R_1 x_1, Q_2 R_2 x_2, Q_3 R_3 x_3, Q_4 R_4 x_4, Q_5 R_5 x_5, Q_6 R_6 x_6, Q_7 R_7 x_7, Q_8 R_8 x_8) = \\ (Q_1^{\blacksquare} R_1^{\blacksquare} x_1, Q_2^{\blacksquare} R_2^{\blacksquare} x_2, Q_3^{\blacksquare} R_3^{\blacksquare} x_3, Q_4^{\blacksquare} R_4^{\blacksquare} x_4, Q_5^{\blacksquare} R_5^{\blacksquare} x_5, Q_6^{\blacksquare} R_6^{\blacksquare} x_6, Q_7^{\blacksquare} R_7^{\blacksquare} x_7, Q_8^{\blacksquare} R_8^{\blacksquare} x_8) \oplus (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8)$$

Using 2.15(i) we have

$$(Q_1 R_1 x_1, Q_2 R_2 x_2, Q_3 R_3 x_3, Q_4 R_4 x_4, Q_5 R_5 x_5, Q_6 R_6 x_6, Q_7 R_7 x_7, Q_8 R_8 x_8) = \\ (Q_1^{\blacksquare} R_1^{\blacksquare} x_1 + b_1, Q_2^{\blacksquare} R_2^{\blacksquare} x_2 + b_2, Q_3^{\blacksquare} R_3^{\blacksquare} x_3 + b_3, Q_4^{\blacksquare} R_4^{\blacksquare} x_4 + b_4, Q_5^{\blacksquare} R_5^{\blacksquare} x_5 + b_5, Q_6^{\blacksquare} R_6^{\blacksquare} x_6 + b_6, Q_7^{\blacksquare} R_7^{\blacksquare} x_7 + b_7, Q_8^{\blacksquare} R_8^{\blacksquare} x_8 + b_8)$$

Using 2.12 we have

$$Q_1 R_1 x_1 = Q_1^{\blacksquare} R_1^{\blacksquare} x_1 + b_1 \\ \Rightarrow x_1 = (Q_1 R_1 - Q_1^{\blacksquare} R_1^{\blacksquare})^{-1} b_1$$

$$Q_2 R_2 x_2 = Q_2^{\blacksquare} R_2^{\blacksquare} x_2 + b_2 \\ \Rightarrow x_2 = (Q_2 R_2 - Q_2^{\blacksquare} R_2^{\blacksquare})^{-1} b_2$$

$$\begin{aligned} Q_3 R_3 x_3 &= Q_3^{\blacksquare} R_3^{\blacksquare} x_3 + b_3 \\ \Rightarrow x_3 &= (Q_3 R_3 - Q_3^{\blacksquare} R_3^{\blacksquare})^{-1} b_3 \\ \\ Q_4 R_4 x_4 &= Q_4^{\blacksquare} R_4^{\blacksquare} x_4 + b_4 \\ \Rightarrow x_4 &= (Q_4 R_4 - Q_4^{\blacksquare} R_4^{\blacksquare})^{-1} b_4 \\ \\ Q_5 R_5 x_5 &= Q_5^{\blacksquare} R_5^{\blacksquare} x_5 + b_5 \\ \Rightarrow x_5 &= (Q_5 R_5 - Q_5^{\blacksquare} R_5^{\blacksquare})^{-1} b_5 \end{aligned} \quad (4.7.3)$$

$$\begin{aligned} Q_6 R_6 x_6 &= Q_6^{\blacksquare} R_6^{\blacksquare} x_6 + b_6 \\ \Rightarrow x_6 &= (Q_6 R_6 - Q_6^{\blacksquare} R_6^{\blacksquare})^{-1} b_6 \end{aligned}$$

$$\begin{aligned} Q_7 R_7 x_7 &= Q_7^{\blacksquare} R_7^{\blacksquare} x_7 + b_7 \\ \Rightarrow x_7 &= (Q_7 R_7 - Q_7^{\blacksquare} R_7^{\blacksquare})^{-1} b_7 \end{aligned}$$

$$\begin{aligned} Q_8 R_8 x_8 &= Q_8^{\blacksquare} R_8^{\blacksquare} x_8 + b_8 \\ \Rightarrow x_8 &= (Q_8 R_8 - Q_8^{\blacksquare} R_8^{\blacksquare})^{-1} b_1 \end{aligned}$$

For solving dual fully fuzzy linear system (3.1) with this method, (for non-square matrix) and in (3.2) apply QR decomposition we have

$$\begin{aligned} A &= Q_1 R_1, B = Q_2 R_2, C = Q_3 R_3, D = Q_4 R_4, \\ E &= Q_5 R_5, F = Q_6 R_6, G = Q_7 R_7, H = Q_8 R_8 \end{aligned}$$

Therefore we have

$$\begin{aligned} Q_1 R_1 x_1 &= b_1 \Rightarrow x_1 = R_1^{-1} Q_1^T b_1 \\ Q_2 R_2 x_2 &= b_2 \Rightarrow x_2 = R_2^{-1} Q_2^T b_2 \\ Q_3 R_3 x_3 &= b_3 \Rightarrow x_3 = R_3^{-1} Q_3^T b_3 \\ Q_4 R_4 x_4 &= b_4 \Rightarrow x_4 = R_4^{-1} Q_4^T b_4 \\ Q_5 R_5 x_5 &= b_5 \Rightarrow x_5 = R_5^{-1} Q_5^T b_5 \\ Q_6 R_6 x_6 &= b_6 \Rightarrow x_6 = R_6^{-1} Q_6^T b_6 \\ Q_7 R_7 x_7 &= b_7 \Rightarrow x_7 = R_7^{-1} Q_7^T b_7 \\ Q_8 R_8 x_8 &= b_8 \Rightarrow x_8 = R_8^{-1} Q_8^T b_8 \end{aligned} \quad (4.7.4)$$

#### 4.8. Linear Programming approach

For solving dual fully fuzzy linear system (3.1) with this method, Consider (3.2), Thus we may write

$$\text{Min } s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7 + s_8$$

Subject to

$$Ax_1 + s_1 = b_1$$

$$Bx_2 + s_2 = b_2$$

$$Cx_3 + s_3 = b_3$$

$$Dx_4 + s_4 = b_4$$

$$Ex_5 + s_5 = b_5$$

$$\begin{aligned} Fx_6 + s_6 &= b_6 \\ Gx_7 + s_7 &= b_7 \\ Hx_8 + s_8 &= b_8 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8 &\geq 0 \end{aligned}$$

#### 4.9. New proposed method for dual fully fuzzy linear system with positive coefficients

To find a solution of dual fully fuzzy linear system  $\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$ , Where the coefficient matrix  $\tilde{A}_1 = (\tilde{a}_{ij})$ ,  $\tilde{A}_2 = (\tilde{b}_{ij})$ ,  $1 \leq i \leq m$ ,  $1 \leq j \leq n$  is a  $m \times n$  fuzzy matrix.

Let  $\tilde{a}_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}, g_{ij}, h_{ij}) \geq 0$ ,  $\tilde{b}_{ij} = (k_{ij}, l_{ij}, m_{ij}, n_{ij}, o_{ij}, z_{ij}, \sigma_{ij}, \rho_{ij}) \geq 0$   
 $\tilde{x} = (x_{1j}, x_{2j}, x_{3j}, x_{4j}, x_{5j}, x_{6j}, x_{7j}, x_{8j}) \geq 0$ ,  $\tilde{b} = (b_{1i}, b_{2i}, b_{3i}, b_{4i}, b_{5i}, b_{6i}, b_{7i}, b_{8i}) \geq 0$  be an octagonal fuzzy numbers then the fully fuzzy linear system can be written as

$$\begin{aligned} \sum_{j=1}^n (a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}, g_{ij}, h_{ij}) \otimes (x_{1j}, x_{2j}, x_{3j}, x_{4j}, x_{5j}, x_{6j}, x_{7j}, x_{8j}) = \\ \sum_{j=1}^n (k_{ij}, l_{ij}, m_{ij}, n_{ij}, o_{ij}, z_{ij}, \sigma_{ij}, \rho_{ij}) \otimes (x_{1j}, x_{2j}, x_{3j}, x_{4j}, x_{5j}, x_{6j}, x_{7j}, x_{8j}) \\ \oplus (b_{1i}, b_{2i}, b_{3i}, b_{4i}, b_{5i}, b_{6i}, b_{7i}, b_{8i}) \quad \forall i=1,2,\dots,m \end{aligned}$$

Using 2.14 (i), (ii and 2.11 we have

$$\begin{aligned} \sum_{j=1}^n (m_{ij}, n_{ij}, p_{ij}, q_{ij}, r_{ij}, s_{ij}, t_{ij}, u_{ij}) = \\ \sum_{j=1}^n (\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}, i_{ij}, j_{ij}, x_{ij}, y_{ij}) \oplus (b_{1i}, b_{2i}, b_{3i}, b_{4i}, b_{5i}, b_{6i}, b_{7i}, b_{8i}), \quad \forall i=1,2,\dots,m \end{aligned}$$

$$\sum_{j=1}^n m_{ij} - \sum_{j=1}^n \alpha_{ij} = b_{1i}, \quad \forall i=1,2,\dots,m$$

$$\sum_{j=1}^n n_{ij} - \sum_{j=1}^n \beta_{ij} = b_{2i}, \quad \forall i=1,2,\dots,m$$

$$\sum_{j=1}^n p_{ij} - \sum_{j=1}^n \gamma_{ij} = b_{3i}, \quad \forall i=1,2,\dots,m$$

$$\sum_{j=1}^n q_{ij} - \sum_{j=1}^n \delta_{ij} = b_{4i}, \quad \forall i=1,2,\dots,m$$

$$\sum_{j=1}^n r_{ij} - \sum_{j=1}^n i_{ij} = b_{5i}, \quad \forall i=1,2,\dots,m$$

$$\sum_{j=1}^n s_{ij} - \sum_{j=1}^n j_{ij} = b_{6i}, \quad \forall i=1,2,\dots,m$$

$$\sum_{j=1}^n t_{ij} - \sum_{j=1}^n x_{ij} = b_{7i}, \quad \forall i=1,2,\dots,m$$

$$\sum_{j=1}^n u_{ij} - \sum_{j=1}^n y_{ij} = b_{8i}, \quad \forall i=1,2,\dots,m$$

The above linear system of equations converts the  $m \times n$  fully fuzzy linear system into  $8m \times 8n$  crisp linear system of equations. The solution of the linear system can be achieved through any one of the classical methods, Schur complements method and also by Linear programming approach.

#### 4.10. Schur Complements

Let  $N$  be an  $n \times n$  matrix written as  $2 \times 2$  block matrix

$$N = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

Where  $P$  is a  $p \times p$  matrix,  $S$  is a  $s \times s$  matrix with  $n = p + s$ ,  $Q$  is a  $p \times s$  matrix and  $R$  is a  $s \times p$  matrix.

To solve the linear system

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\begin{aligned} Px + Qy &= c \\ Rx + Sy &= d \end{aligned}$$

Assume that  $S$  is invertible

$$y = S^{-1}(d - Rx)$$

$$Px + Q[S^{-1}(d - Rx)] = c$$

$$(P - QS^{-1}R)x = c - QS^{-1}d$$

If the matrix  $(P - QS^{-1}R)$  is invertible then  $x = (P - QS^{-1}R)^{-1} [c - QS^{-1}d]$

$$y = S^{-1}\{d - R(P - QS^{-1}R)^{-1} [c - QS^{-1}d]\}$$

The matrix  $(P - QS^{-1}R)$  is called the schur complement of  $S$  in  $N$ .

### 5. NUMERICAL EXAMPLE

#### 5.1. Solve the following dual fully fuzzy linear system

$$(14,16,18,20,22,24,26,28) \otimes \tilde{x} \oplus (15,18,21,24,27,30,33,36) \otimes \tilde{y} = (13,14,15,16,17,18,19,20) \otimes \tilde{x} \oplus (14,15,16,17,18,19,20,21) \otimes \tilde{y} \oplus (5,20,43,74,113,160,215,278)$$

$$\begin{aligned} (20,24,28,32,36,40,44,48) \otimes \tilde{x} \oplus (21,23,25,27,29,31,33,35) \otimes \tilde{y} = \\ (14,17,20,23,26,29,32,35) \otimes \tilde{x} \oplus (10,11,12,13,14,15,16,17) \otimes \tilde{y} \oplus \\ (45,76,113,156,205,260,321,388) \end{aligned}$$

Solution:

$$(14,16,18,20,22,24,26,28) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (15,18,21,24,27,30,33,36) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = (13,14,15,16,17,18,19,20) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (14,15,16,17,18,19,20,21) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \oplus (5,20,43,74,113,160,215,278)$$

$$(20,24,28,32,36,40,44,48) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (21,23,25,27,29,31,33,35) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = (14,17,20,23,26,29,32,35) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (10,11,12,13,14,15,16,17) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \oplus (45,76,113,156,205,260,321,388)$$

Using algorithms given in 4.1, 4.2, 4.3, 4.4, 4.5 we have

$$\tilde{x} = (2,4,6,8,10,12,14,16) \quad \tilde{y} = (3,4,5,6,7,8,9,10)$$

### 5.2. Solve the following dual fully fuzzy linear system Cholesky decomposition

$$(14,16,18,20,22,24,26,28) \otimes \tilde{x} \oplus (15,18,21,24,27,30,33,36) \otimes \tilde{y} = (13,14,15,16,17,18,19,20) \otimes \tilde{x} \oplus (14,15,16,17,18,19,20,21) \otimes \tilde{y} \oplus (5,20,43,74,113,160,215,278)$$

$$(15,18,21,24,27,30,33,36) \otimes \tilde{x} \oplus (21,23,25,27,29,31,33,35) \otimes \tilde{y} = (14,15,16,17,18,19,20,21) \otimes \tilde{x} \oplus (10,11,12,13,14,15,16,17) \otimes \tilde{y} \oplus (35,60,95,140,195,260,335,420)$$

Solution:

$$(14,16,18,20,22,24,26,28) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (15,18,21,24,27,30,33,36) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = (13,14,15,16,17,18,19,20) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (14,15,16,17,18,19,20,21) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \oplus (5,20,43,74,113,160,215,278)$$

$$(15,18,21,24,27,30,33,36) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (21,23,25,27,29,31,33,35) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = (14,15,16,17,18,19,20,21) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (10,11,12,13,14,15,16,17) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \oplus (35,60,95,140,195,260,335,420)$$

Solution: Using algorithm given in 4.6 we have

$$\tilde{x} = (2,4,6,8,10,12,14,16) \quad \tilde{y} = (3,4,5,6,7,8,9,10)$$

### 5.3. Solve the following dual fully fuzzy linear system QR decomposition method

$$(14,16,18,20,22,24,26,28) \otimes \tilde{x} \oplus (15,18,21,24,27,30,33,36) \otimes \tilde{y} = (13,14,15,16,17,18,19,20) \otimes \tilde{x} \oplus (14,15,16,17,18,19,20,21) \otimes \tilde{y} \oplus (5,20,43,74,113,160,215,278)$$

$$(20,24,28,32,36,40,44,48) \otimes \tilde{x} \oplus (21,23,25,27,29,31,33,35) \otimes \tilde{y} = (14,17,20,23,26,29,32,35) \otimes \tilde{x} \oplus (10,11,12,13,14,15,16,17) \otimes \tilde{y} \oplus (45,76,113,156,205,260,321,388)$$

$$(15,18,21,24,27,30,33,36) \otimes \tilde{x} \oplus (21,23,25,27,29,31,33,35) \otimes \tilde{y} = 14,15,16,17,18,19,20,21) \otimes \tilde{x} \oplus (10,11,12,13,14,15,16,17) \otimes \tilde{y} \oplus (35,60,95,140,195,260,335,420)$$

Solution:

$$(14,16,18,20,22,24,26,28) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (15,18,21,24,27,30,33,36) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = (13,14,15,16,17,18,19,20) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (14,15,16,17,18,19,20,21) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \oplus (5,20,43,74,113,160,215,278)$$

$$(20,24,28,32,36,40,44,48) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (21,23,25,27,29,31,33,35) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = (14,17,20,23,26,29,32,35) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (10,11,12,13,14,15,16,17) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \oplus (45,76,113,156,205,260,321,388)$$

$$(15,18,21,24,27,30,33,36) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (21,23,25,27,29,31,33,35) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = (14,15,16,17,18,19,20,21) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (10,11,12,13,14,15,16,17) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \oplus (35,60,95,140,195,260,335,420)$$

Using algorithm given in 4.7 we have

$$\tilde{x} = (2,4,6,8,10,12,14,16) \quad \tilde{y} = (3,4,5,6,7,8,9,10)$$

#### 5.4. Solve the following dual fully fuzzy linear system

$$(14,16,18,20,22,24,26,28) \otimes \tilde{x} \oplus (15,18,21,24,27,30,33,36) \otimes \tilde{y} = (13,14,15,16,17,18,19,20) \otimes \tilde{x} \oplus (14,15,16,17,18,19,20,21) \otimes \tilde{y} \oplus (5,20,43,74,113,160,215,278)$$

$$(20,24,28,32,36,40,44,48) \otimes \tilde{x} \oplus (21,23,25,27,29,31,33,35) \otimes \tilde{y} = 14,17,20,23,26,29,32,35) \otimes \tilde{x} \oplus (10,11,12,13,14,15,16,17) \otimes \tilde{y} \oplus (45,76,113,156,205,260,321,388)$$

Solution:

$$(14,16,18,20,22,24,26,28) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (15,18,21,24,27,30,33,36) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = (13,14,15,16,17,18,19,20) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (14,15,16,17,18,19,20,21) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \oplus (5,20,43,74,113,160,215,278)$$

$$(20,24,28,32,36,40,44,48) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (21,23,25,27,29,31,33,35) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = (14,17,20,23,26,29,32,35) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (10,11,12,13,14,15,16,17) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \oplus (45,76,113,156,205,260,321,388)$$

Using algorithm given in 4.9 we have

$$\begin{aligned} &(14x_1, 16x_2, 18x_3, 20x_4, 22x_5, 24x_6, 26x_7, 28x_8) \oplus \\ &(15y_1, 18y_2, 21y_3, 24y_4, 27y_5, 30y_6, 33y_7, 35y_8) = \\ &(13x_1, 14x_2, 15x_3, 16x_4, 17x_5, 18x_6, 19x_7, 20x_8) \oplus \\ &(14y_1, 15y_2, 16y_3, 17y_4, 18y_5, 19y_6, 20y_7, 21y_8) \oplus \\ &(5, 20, 43, 74, 113, 160, 215, 278) \end{aligned}$$

$$\begin{aligned} &(20x_1, 24x_2, 28x_3, 32x_4, 36x_5, 40x_6, 44x_7, 48x_8) \oplus \\ &(21y_1, 23y_2, 25y_3, 27y_4, 29y_5, 31y_6, 33y_7, 35y_8) = \\ &(14x_1, 17x_2, 20x_3, 23x_4, 26x_5, 29x_6, 32x_7, 35x_8) \oplus \\ &(10y_1, 11y_2, 12y_3, 13y_4, 14y_5, 15y_6, 16y_7, 17y_8) \oplus \\ &(45, 76, 113, 156, 205, 260, 321, 388) \end{aligned}$$

$$\begin{aligned} 14x_1 + 15y_1 &= 13x_1 + 14y_1 + 5; 16x_2 + 18y_2 = 14x_2 + 15y_2 + 20; \\ 18x_3 + 21y_3 &= 15x_3 + 16y_3 + 43; 20x_4 + 24y_4 = 16x_4 + 17y_4 + 74; \\ 22x_5 + 27y_5 &= 17x_5 + 18y_5 + 113; 24x_6 + 30y_6 = 18x_6 + 19y_6 + 160; \\ 26x_7 + 33y_7 &= 19x_7 + 20y_7 + 215; 28x_8 + 35y_8 = 20x_8 + 21y_8 + 278; \end{aligned}$$

$$\begin{aligned} 20x_1 + 21y_1 &= 14x_1 + 10y_1 + 45; 24x_2 + 23y_2 = 17x_2 + 11y_2 + 76; \\ 28x_3 + 25y_3 &= 20x_3 + 12y_3 + 113; 32x_4 + 27y_4 = 23x_4 + 13y_4 + 156; \\ 36x_5 + 29y_5 &= 26x_5 + 14y_5 + 205; 40x_6 + 31y_6 = 29x_6 + 15y_6 + 260; \\ 44x_7 + 33y_7 &= 32x_7 + 16y_7 + 321; 48x_8 + 35y_8 = 35x_8 + 17y_8 + 388 \end{aligned}$$

$$\begin{aligned} x_1 + y_1 &= 5; 2x_2 + 3y_2 = 20; 3x_3 + 5y_3 = 43; 4x_4 + 7y_4 = 74; 5x_5 + 9y_5 = 113; \\ 6x_6 + 11y_6 &= 160; 7x_7 + 13y_7 = 215; 8x_8 + 15y_8 = 278; 6x_1 + 11y_1 = 45; \\ 7x_2 + 12y_2 &= 76; 8x_3 + 13y_3 = 113; 9x_4 + 14y_4 = 156; 10x_5 + 15y_5 = 205; \\ 11x_6 + 16y_6 &= 260; 12x_7 + 17y_7 = 321; 13x_8 + 18y_8 = 388 \end{aligned}$$

Now the above 16 x 16 linear system of equations can be solved Using any one of the classical method, as well as using 4.10. Schur Complements method we have (We used TORA software)

$$\tilde{x} = (2, 4, 6, 8, 10, 12, 14, 16) \quad \tilde{y} = (3, 4, 5, 6, 7, 8, 9, 10)$$

### 5.5. Solve the following dual fully fuzzy linear system

$$(14, 16, 18, 20, 22, 24, 26, 28) \otimes \tilde{x} \oplus (15, 18, 21, 24, 27, 30, 33, 36) \otimes \tilde{y} = (13, 14, 15, 16, 17, 18, 19, 20) \otimes \tilde{x} \oplus (14, 15, 16, 17, 18, 19, 20, 21) \otimes \tilde{y} \oplus (5, 20, 43, 74, 113, 160, 215, 278)$$

$$(20, 24, 28, 32, 36, 40, 44, 48) \otimes \tilde{x} \oplus (21, 23, 25, 27, 29, 31, 33, 35) \otimes \tilde{y} = 14, 17, 20, 23, 26, 29, 32, 35) \otimes \tilde{x} \oplus (10, 11, 12, 13, 14, 15, 16, 17) \otimes \tilde{y} \oplus (45, 76, 113, 156, 205, 260, 321, 388)$$

Solution:

$$\begin{aligned} &(14, 16, 18, 20, 22, 24, 26, 28) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (15, 18, 21, 24, 27, 30, 33, 36) \otimes \\ &(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = (13, 14, 15, 16, 17, 18, 19, 20) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus \\ &(14, 15, 16, 17, 18, 19, 20, 21) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \oplus (5, 20, 43, 74, 113, 160, 215, 278) \end{aligned}$$

$$(20,24,28,32,36,40,44,48) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (21,23,25,27,29,31,33,35) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) = (14,17,20,23,26,29,32,35) \otimes (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \oplus (10,11,12,13,14,15,16,17) \otimes (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \oplus (45,76,113,156,205,260,321,388)$$

Using algorithm given in 4.9 we have

$$(14x_1, 16x_2, 18x_3, 20x_4, 22x_5, 24x_6, 26x_7, 28x_8) \oplus (15y_1, 18y_2, 21y_3, 24y_4, 27y_5, 30y_6, 33y_7, 35y_8) = (13x_1, 14x_2, 15x_3, 16x_4, 17x_5, 18x_6, 19x_7, 20x_8) \oplus (14y_1, 15y_2, 16y_3, 17y_4, 18y_5, 19y_6, 20y_7, 21y_8) \oplus (5, 20, 43, 74, 113, 160, 215, 278)$$

$$(20x_1, 24x_2, 28x_3, 32x_4, 36x_5, 40x_6, 44x_7, 48x_8) \oplus (21y_1, 23y_2, 25y_3, 27y_4, 29y_5, 31y_6, 33y_7, 35y_8) = (14x_1, 17x_2, 20x_3, 23x_4, 26x_5, 29x_6, 32x_7, 35x_8) \oplus (10y_1, 11y_2, 12y_3, 13y_4, 14y_5, 15y_6, 16y_7, 17y_8) \oplus (45, 76, 113, 156, 205, 260, 321, 388)$$

$$14x_1 + 15y_1 = 13x_1 + 14y_1 + 5; 16x_2 + 18y_2 = 14x_2 + 15y_2 + 20; \\ 18x_3 + 21y_3 = 15x_3 + 16y_3 + 43; 20x_4 + 24y_4 = 16x_4 + 17y_4 + 74; \\ 22x_5 + 27y_5 = 17x_5 + 18y_5 + 113; 24x_6 + 30y_6 = 18x_6 + 19y_6 + 160; \\ 26x_7 + 33y_7 = 19x_7 + 20y_7 + 215; 28x_8 + 35y_8 = 20x_8 + 21y_8 + 278;$$

$$20x_1 + 21y_1 = 14x_1 + 10y_1 + 45; 24x_2 + 23y_2 = 17x_2 + 11y_2 + 76; \\ 28x_3 + 25y_3 = 20x_3 + 12y_3 + 113; 32x_4 + 27y_4 = 23x_4 + 13y_4 + 156; \\ 36x_5 + 29y_5 = 26x_5 + 14y_5 + 205; 40x_6 + 31y_6 = 29x_6 + 15y_6 + 260 \\ 44x_7 + 33y_7 = 32x_7 + 16y_7 + 321; 48x_8 + 35y_8 = 35x_8 + 17y_8 + 388$$

$$x_1 + y_1 = 5; 2x_2 + 3y_2 = 20; 3x_3 + 5y_3 = 43; 4x_4 + 7y_4 = 74; 5x_5 + 9y_5 = 113; \\ 6x_6 + 11y_6 = 160; 7x_7 + 13y_7 = 215; 8x_8 + 15y_8 = 278;$$

$$6x_1 + 11y_1 = 45; 7x_2 + 12y_2 = 76; 8x_3 + 13y_3 = 113; 9x_4 + 14y_4 = 156; \\ 10x_5 + 15y_5 = 205; 11x_6 + 16y_6 = 260; 12x_7 + 17y_7 = 321; 13x_8 + 18y_8 = 388$$

Now the above linear system can be solved by using two phase method

Minimize  $(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_8 + r_9 + r_{10} + r_{11} + r_{12} + r_{13} + r_{14} + r_{15} + r_{16})$   
Subject to

$$x_1 + y_1 + r_1 = 5; 2x_2 + 3y_2 + r_2 = 20; 3x_3 + 5y_3 + r_3 = 43; 4x_4 + 7y_4 + r_4 = 74; \\ 5x_5 + 9y_5 + r_5 = 113; 6x_6 + 11y_6 + r_6 = 160; 7x_7 + 13y_7 + r_7 = 215; \\ 8x_8 + 15y_8 + r_8 = 278; 6x_1 + 11y_1 + r_9 = 45; 7x_2 + 12y_2 + r_{10} = 76; \\ 8x_3 + 13y_3 + r_{11} = 113; 9x_4 + 14y_4 + r_{12} = 156; 10x_5 + 15y_5 + r_{13} = 205; \\ 11x_6 + 16y_6 + r_{14} = 260; 12x_7 + 17y_7 + r_{15} = 321; 13x_8 + 18y_8 + r_{16} = 388$$

$$x_{i+1} - x_i \geq 0, 1 \leq i \leq 8, y_{i+1} - y_i \geq 0, 1 \leq j \leq 8$$

Where  $r_i, 1 \leq i \leq 16$  are artificial variables

Using Tora software we have  $\tilde{x} = (2,4,6,8,10,12,14,16)$   $\tilde{y} = (3,4,5,6,7,8,9,10)$

## CONCLUSION

In this paper, the solution of dual fully fuzzy linear system of the form  $\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \oplus \tilde{b}$ , whose coefficients are octagonal fuzzy numbers is obtained by Matrix inversion, Cramer's rule, LU,  $LL^T$ , QR decomposition methods, direct methods and indirect methods. These methods are useful when the system is square as well as non square.

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