CAUCHY-RIEMMAN SUBMANIFOLDS OF COMPLEX FINSLER MANIFOLDS

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ABSTRACT:

   Andreotti and Hill (1972) have studied complex characteristic coordinate and tangential Cauchy-Riemman equations. Bejancu (1987) has studied complex finsler spaces and C.R. structures. Further, Farinola (1988) has studied a characterization of the Kaehler condition on Finsler spaces also, Dragomir (1989) has studied Cauchy-rieman submanifolds of Kaehlerian Finsler Spaces. In this paper, we have studied Cauchy-Riemman submanifolds of complex Finsler manifolds and several theorem have been derived .

KEY WORDS: complex finsler manifolds chern (c.l.c), $\eta -$ Einstein spaces, constant curvature.

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1. INTRODUCTION.

   Let (M^{2n}, L, J) be a Kaehlerian Finsler space Dragomir and Ianus(1983), and $(\pi^{-1}TM^{2n}, g)$ its induced Riemannian bundle By a recent result Farnola(1988), the torsion tensor N_j^v vanishes. Therefore we may apply Bejancu (1987) such as to conclude that $V(M^{2n}) = T(M^{2n})\backslash 0$ admits a (naturally induced) Cauchy-Riemann (CR) structure H^v , i.e., a complex subbundle II^v of the complexified tangent bundle

$$T^c\big(V(M^{2n})\big) = T\big(V(M^{2n})\big)\otimes C,$$

Such that i) H^v is involutive, ii) $H^v \cap \overline{H^v} = (0), and$ iii) $*R\,(H^v) = \ker(d\pi),$ where $\pi: V(M^{2n}) \to M^{2n}$ is the natural projection. On the other hand, let $\tilde{J}$ be the lift of J (with respect to the nonlinear connection of the Cartan connection of (M^{2n}, L)) cf. our (3.4). Then $\tilde{J}$ is naturally extended (by C-linearity) to

$T^c\big(V(M^{2n})\big)$; let $T^{1,0}\big(V(M^{2n})\big)$ be the bundle of all eigenvalue $i = \sqrt{-1}$ of $\tilde{J}$. leaving definitions momentarily aside, we formulate the following definition given by Dragomir (1989)

Definition (1.1): Let (M^{2n}, L, J) be a Kaehlerian Finsler space whose Cartan connection has an integrable $(R^i_{jk} = 0)$ associated nonlinear connection N on $V(M^{2n})$. Then $V(M^{2n})$ admits a Cauchy-Riemann *R (H^h) such that $* R(H^h) = N$ and

(1.1) $T^{1,0}\big(V(M^{2n})\big) = H^h \oplus H^v.$

Definition (1.2): Let M^m be an invariant , i.e.

$$j_u(\pi_u^{-1} T M^m) = \pi_u^{-1} T M^m, \qquad\qquad u \in V(M^m),$$

Submanifold of the complex v-space form $(M^{2n}(c), L, J), n \geq 2$. Then M^m is a complex v-space form (with respect to the induced connection) of the same holomorphic v-sectional curvature c if and only if Q=0, i.e. then vertical second fundamental form vanishes.

Definition (1.3): Let M^m be an invariant totally-geodesic submanifold of the Kaehlerian Finsler   space (M^{2n}, L, J) . Then M^m is h-minimal.

.   **Definition (1.4):** Let M^n be a totally-real, i.e.

$$J_u(\pi_u^{-1} T M^n) = E(\psi)_u , \qquad u \in v(M^n),$$

Submanifold of the Kaehlerian Finsler space (M^{2n}, L, J). If M^n has a v-flat normal connection (i.e. $S^\perp = 0$ then M^n is a Berwald-Cartan space.

Definition (1.5): Let M^n be a totally-real submanifold of the Kachlerian Finsler space (M^{2n}, L, J), having an h-flat normal connection. Then the induced connection of M^n has a vanishing horizontal curvature tensor, i.e R=0.

3. COMPLEX FINSLER STRUCTURES.

Let (M^{2n}, L), $n \geq 1$, be a real 2n-dimensional Finsler space with the Lagrangian function L: T(M^{2n}) [0, +∞]. Here T (M^{2n}) → M^{2n} denotes the tangent bundle over M^{2n}. Let j: M^{2n} → T(M^{2n}) be the natural imbedding , i.e.

$$j(x) = 0 \in T_x(M^{2n}), \qquad x \in M^{2n}.$$

We put $\qquad V(M^{2n}) = T(M^{2n}) \backslash j(M^{2n}).$

Let $\pi: V(M^{2n}) \to M^{2n}$ be the natural projection and $\pi^{-1}TM^{2n} \to V(M^{2n})$ the pullback bundle of $T(M^{2n})$ by π. A bundle morphism $J: \pi^{-1}TM^{2n} \to \pi^{-1}TM^{2n}$, $J^2 = -I$ is said to ba a Finslerian almost complex structure on M^{2n}.

Let $u \in V(M^{2n})$; then $\pi_u^{-1}TM^{2n} = \{u\} \times T_x(M^{2n})$, $x = \pi(u)$, denotes the fiber over u in $\pi^{-1}TM^{2n}$. Any ordinary almost complex structure J: $T(M^{2n}) \to T(M^{2n})$, $J^2 = -1$, admits a natural lift to a Finslerian almost complex structure $\bar{J}$ given by

$$J_n X = (u, J_x \hat{\pi} X), \quad x = \pi(u), \quad X \in \pi_u^{-1}TM^{2n}, u \in V(M^{2n}).$$

Here $\hat{\pi}$ denotes the projection onto the second factor of the product manifolds $V(M^{2n}) \times T(M^{2n})$.

Let (U, x^i) be a system of local coordinates on M^{2n} and let $(\pi^{-1}(U), x^i, y^i)$ be the naturally induced local coordinates on $V(M^{2n})$ The vertical lift is the bundle isomorphism $\gamma: \pi^{-1}TM^{2n} \to \ker(d\pi)$ defined by

$$\gamma X_i = \dot{\partial}_i, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i}.$$

Clearly, the definition of γ does not depend upon the choice of local coordinates.

Here $\qquad X_i(u) = \left(u, \frac{\partial}{\partial x^i} \otimes_{\pi(u)}\right), \quad u \in \pi^{-1}(U), \quad 1 \le i \le 2n.$

To make this construction precise, let us note that any tangent vector field X on M^{2n} admits a natural lift X to a cross-section of $\pi^{-1}TM^{2n}$ defined by

$$\bar{X}(u) = \left(u, X(\pi(u))\right), \quad u \in V(M^{2n}).$$

Note that X_i are the natural lifts of the (locally defined) tangent vector fields $\partial/\partial x^i$. Clearly $\{X_i\}_{1 \le i \le 2n}$ form a (local) frame of $\pi^{-1}TM^{2n}$

Let

$$g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2.$$

We put

$$g_u(X,Y) = g_{ij}(u)X^i Y^j, \quad X = X^i X_i, \quad Y = Y^i X_i, \quad u \in \pi^{-1}(U),$$

for any cross-section X, Y in $\pi^{-1}TM^{2n}$. The definition of g_u does not depend upon the choice of local coordinates around $x = \pi(u), u \in M^{2n}$.   Since (M^{2n}, L) is a Finsler space, for each $u \in \pi^{-1}(U)$ the quadratic form $g_{ij}(u)\xi^i \xi^j$ is positive definite. Therefore $\pi^{-1}TM^{2n}$ turns into a Riemannian vector bundle with the Riemann (bundle) metric $g: u \to g_u$.   Then $(\pi^{-1}TM^{2n}, g)$ is called induced bundle of the Finsler manifold (M^{2n}, L). Cross-section in the induced bundle is refered to as Finsler vector fields on M^{2n}.

Let J be a Finslerian almost complex structure on M^{2n}. Since the induced bundle and the vertical bundle ker $(d\pi)(over\ V(M^{2n})$ are isomorphic, ker $(d\pi)$ turns into a Riemannian bundle in a natural way, with the metric $g^v(Z,W) = g(\gamma^{-1}Z, \gamma^{-1}W)$ for any vertical tangent vector fields Z, W on (M^{2n}). Moreover $ker(d\pi)$ is a complex vector bundle since each fiber $ker(d_u\pi)$ carries the complex structure j_u^v defined by

$$j_u^v = \gamma_u \circ J_u \circ \gamma_u^{-1}, \quad u \in V(M^{2n}).$$

Let $[J^v, J^v]$ be the torsion of the (1,1)-tensor fields J^v. This is $ker(d\pi)$-valued, since $ker(d\pi)$ is involutive. Let $JX_i = J_i^j(x,y)X_j$.   Then $J_j^i J_k^j = -\delta_k^i$.

We define $N_j^v(X_i, X_j) = N_{ij}^K X_k$ where:

(2.1) $$N_{jk}^i = J_j^m \frac{\partial J_k^i}{\partial y^m} - J_k^m \frac{\partial J_j^i}{\partial y^m} + J_i^m \frac{\partial J_k^m}{\partial y^j} - J_m^i \frac{\partial J_j^m}{\partial y^k}.$$

Clearly the definition of N_j^v does not depend upon the choice of local coordinates. i.e.

(2.2) $$\gamma N_j^v(X,Y) = [J^v, J^v](\gamma X, \gamma Y)$$

for any Finsler vector fields X , Y on M^{2n} . Bejancu (1987) given by if $N_j^v = 0$.

then J may be called a complex Finsler structure. Nevertheless, this concept is not entirely satisfactory since, in this sense, the natural life of any (not necessarily integrable) almost complex structure on M^{2n} is a complex Finsler structure, by (2.1).

A differentiable 2n-distribution $N: u \mapsto N_u$ on $V(M^{2n})$ is called a nonlinear connection on $V(M^{2n})$ if each N_u is a direct summand to the vertical space $ker(d_u\pi)$ in $T_u(V(M^{2n})), \; u \in V(M^{2n})$. In classical tensor notation N might be represented by the *Pfaffian* system:

(2.3) $$dy^i + N_j^i dx^j = 0.$$

Here $N_j^i \in C^\infty(\pi^{-1}(U))$ are the coefficients of the nonlinear connection; that is, if $u \in \pi^{-1}(U)$, then N_u is spanned by the tangent vectors

$$\frac{\delta}{\delta x^i} = \partial_i - N_i^j \dot\partial_j, \quad \partial_i = \frac{\partial}{\partial x^i}.$$

Let N be a nonlinear connection on $V(M^{2n})$. We shall the bundle epimorphism $L : T(V(M^{2n})) \to \pi^{-1}TM^{2n}$, defined by

$$L_u Z = (u, (d_u\pi)z), \quad Z \in T_u(V(M^{2n})), \qquad u \in V(M^{2n}).$$

Clearly, the restriction of L_u to N_u is a $\mathbb{R}$-linear isomorphism $N_u \approx \pi_u^{-1}TM^{2n}$. Let β_u denote its inverse. The resulting bundle isomorphism $\beta: \pi^{-1}TM^{2n} \to N$ is refered to as the horizontal lift (with respect to the nonlinear connection N). If J is a Finslerian almost complex structure on M^{2n} then we may put $J^h = \beta \circ J \circ L$ such as to define a complex structure on the bundle $N \to V(M^{2n})$.

We shall need the Dombrowski mapping, i.e. the bundle morphism

$$G: T(V(M^{2n})) \to \pi^{-1}TM^{2n}, \qquad GZ = \gamma^{-1}Z_v ,$$

Where Z_v denotes the vertical part of Z with repect to the direct to the sum decomposition

$$T_u(V(M^{2n})) = N_u \oplus ker(d_u\pi), \quad u \in V(M^{2n}).$$

With any Finslerian almost complex structure J one may associate an almost complex structure $\tilde{J}$ on $V(M^{2n})$ defined by:

(2.4) $\qquad \tilde{J} = \beta \circ J \circ L + \gamma \circ J \circ G.$

Note that the restriction of J_u to N_u , respectively to $ker\,(d_u \pi)$, coincides with J_u^h , respectively with J_u^v, for any $u \in V(M^{2n})$.

Let (M^{2n}, L) be a Finsler manifold and $(\pi^{-1} TM^{2n}, g)$ its induced bundle. Let N be a fixed nonlinear connection on $V(M^{2n})$. The Sasaki metric $\tilde{g}$ is given by:

(2.5) $\qquad \tilde{g}(Z, W) = g(LZ, LW) + g(GZ, GW).$

Thus $V(M^{2n})$ turns naturally into a (noncompact) Riemannian manifolds. Moreover, $(V(M^{2n}), \tilde{g}, \tilde{J})$ is well known to be almost Hermitian.

We shall need the torsion $N_J^h(X_i, X_j) = A_{ij}^k X_k$ where:

(2.6) $\qquad A_{ij}^k = J_j^m \dfrac{\delta J_k^i}{\delta x^m} - J_k^m \dfrac{\delta J_j^i}{\delta x^m} + J_m^i \dfrac{\delta J_k^m}{\delta x^j} - J_i^m \dfrac{\delta J_j^m}{\delta x^k}.$

Clearly the definition of N_J^h does not depend upon the choice of local coordinates. Therefore we have

(2.7) $\qquad N_J^h(X, Y) = L[J^h, J^h](\beta X, \beta Y)$

For any Finsler vector fields X, Y on M^{2n}.

3. KAEHLERIAN FINSLER MANIFOLDS.

Let (M^{2n}, L) be a Finsler manifold endowed with the Finslerian almost complex structure J. Then (M^{2n}, L, J) is said to be an almost Hermitian Finsler manifold if $g(J, X, J, Y) = g(X, Y)$, for any Finsler vector fields X, Y on $V(M^{2n})$. Let ∇ be a connection in the induced bundle $(\pi^{-1} TM^{2n}, g)$. It is said to be metrical if $\nabla_g = 0,$ respectively almost complex if $\nabla J = 0$. A tangent vector field Z on $V(M^{2n})$ is horizontal (with respect to ∇) if $\nabla_z v = 0$. Here v denotes the Liouville vector field, i.e. the cross section in the induced bundle defined by

$$v(u) = (u, u), u \in V(M^{2n}) .$$

Let N be the distribution of all horizontal tangent vectors on (M^{2n}) ; it is refered to as the horizontal distribution of ∇ . Then ∇ is regular if its horizontal distribution N is a nonlinear connection on $V(M^{2n})$.   A pair (∇, N) consisting of a connection in $\pi^{-1}TM^{2n}$ and a nonlinear connection on $V(M^{2n})$ is called a Finsler connection on M^{2n} . Here we have any regular connection in $\pi^{-1}TM^{2n}$ gives raise to a Finsler connection on M^{2n} .

Let (∇, N) be a Finsler connection; then two concept of torsion tensor fields are usually associated with (∇, N) , namely

$$\tilde{T}(Z,W) = \nabla_z L\,W - \nabla_w L\,Z - L[Z,W],$$

$$\tilde{T}_1(Z,W) = \nabla_z G\,W - \nabla_w G\,Z - G[Z,W],$$

For any tangent vector fields Z, W on $V(M^{2n})$. Let also $\tilde{R}$ denote the curvature 2-form of ∇ . Several fragments of $\tilde{T}$, $\tilde{T}_1$ and $\tilde{R}$ are usually derived by means of the bundle morphisms i.e.

$$T(X,Y) = \tilde{T}(\beta X, \beta Y), \qquad C(X,Y) = \tilde{T}(\gamma X, \beta Y),$$

$$R^1(X,Y) = \tilde{T}_1(\beta X, \beta Y), \qquad P^1(X,Y) = \tilde{T}_1(\gamma X, \beta Y),$$

$$S^1(X,Y) = \tilde{T}_1(\gamma X, \gamma Y), \qquad R(X,Y)Z = \tilde{R}(\beta X, \beta Y)Z,$$

$$P(X,Y)Z = \tilde{R}(\gamma X, \beta Y)Z, \qquad S(X,Y)Z = \tilde{R}(\gamma X, \gamma Y)Z.$$

We may define no 'vertical' component of $\tilde{T}$ since clearly $\tilde{T}(\gamma X, \gamma Y) = 0$. For any Finsler vector fields X, Y on M^{2n} .

Therefore we have,

(3.1) $\gamma R^1(X,Y) = [\beta X, \beta Y],$

i.e. R^1 is the obstruction towards the integrability on N. In spite of being defined in terms of ∇ the torsion R^1 depends essentially on N only, as easily seen in local coordinates, i.e.

$$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j},$$

Where $R^1(X_i, X_j) = R_{ij}^k X_k$.

The fundamental theorem of Finsler geometry asserts that there exists a unique regular connection ∇ in the induced bundle $(\pi^{-1}TM^{2n}, g)$ of the given Finsler manifolds (M^{2n}, L).

Such that (i) ∇ is metrical, (ii) T=0, $S^1 = 0$. It is refered to as the Cartan connection of (M^{2n}, L). Then (M^{2n}, L, J) is called a Kaehlerian Finsler manifolds if its Cartan connection is almost   complex given by Dragomir and Ianus (1982).

4. CAUCHY-RIEMANN STRUCTURE ON THE TANGENT BUNDLE

If N is an arbitrary C^∞-manifolds and $T^c(N) = T(N) \otimes C$ denotes the complexification of its tangent bundle, then a Cauchy-Riemann (CR) structure on N is a complex subbundle H of $T^c(N)$.

Such that   (i) $H \cap \bar{H} = 0$ (ii) H is involutive . Here a bar denotes complex conjugation.

Let J be a Finslerian almost complex structure on M^{2n}. Let N be a nonlinear connection on $V(M^{2n})$. If $N_j^h = 0$, we consider the   C-vector subbundle H^h of $T^c(V(M^{2n}))$ defined by $\mapsto H_u^h$, where H_u^h consists of all complex tangent vectors $X \otimes 1 - J^h X \otimes i,\ i = \sqrt{-1},\ X \in N_u,\ u \in V(M^{2n})$.

Suppose $R^1 = 0$. By (2.7) and (3.1), it follows that $[J^h, J^h] = 0$. Consequently H^h is involutive. Also, by the definition of H^h , one has *R $(H_u) = N_u,\ u \in V(M^{2n})$. That is $V(M^{2n})$ turns into a CR manifold.

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