Marangoni Instability in a Liquid Layer with Insulating Free Slip Bottom

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Abstract

The onset of cellular convection induced by surface tension gradients on a horizontal layer of liquid heated from below for insulating lower boundary with free-slip condition is examined using classical linear stability analysis. It is established that the critical Marangoni number decreases along with the critical wave number. The nature of the neutral state is examined by solving the characteristic value equation for the underlying problem numerically. It is shown that the neutral state is stationary rather than an oscillatory one.

Keywords: *Free-slip, insulating, linear stability, stationary, surface tension.*

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Nomenclature

d	–	*depth of the liquid layer in the unperturbed state,*
w	–	*z-component of velocity perturbation,*
θ	–	*z-component of temperature perturbation,*
T_0	–	*temperature of the lower boundary surface,*
T_1	–	*temperature of the upper boundary surface,*
ρ	–	*density,*
ν	–	*kinematic viscosity,*
κ	–	*thermal diffusivity,*
q	–	*heat transfer coefficient between the upper free surface and the air phase,*
σ	–	*rate of change of surface tension with temperature evaluated at temperature* T_1,
$\sqrt{a_x^2 + a_y^2}$	–	*resultant wave number.*

1.1 Introduction

The problem of the onset of convective instability in a thin layer of fluid heated from below has its origin in the experimental observations of Bénard [1, 2]. Rayleigh [3] gave the first analytical treatment of the problem and explained the phenomenon in terms of buoyancy arising due to expansion of a heated liquid and that what decides the stability or otherwise of the problem depends upon the numerical value of the non-dimensional number R called the Rayleigh number. Further contributions made by many authors are discussed in the monographs by Chandrasekhar [4]. All these explanations in terms of buoyancy may be sufficient in the case of a liquid layer thicker than about 1 cm or when there is no free surface. Block [5] pointed out in his brief report that Bénard convection in a liquid film with a free surface could be produced by surface tension tractions arising due to variation of temperature. Pearson [6], who neglected buoyancy, and following small disturbance analysis analogous to that carried out by Rayleigh and others for the case of buoyancy driven instability, gave the first theoretical explanation of the phenomenon in terms of surface tension that depends upon the non-dimensional number M called the

Marangoni number. He showed that surface tension forces can cause convection when the Marangoni number exceeds a critical value. Pearson [6] obtained the critical Marangoni number M_c = 79.607 and the critical wave number a_c = 1.9929 for the conducting case, and M_c = 48 and a_c is zero for the insulating case of the lower rigid boundary . His analysis has been extended by many authors (for example, Scriven and Sternling [7], Smith [8], Takashima [9, 10, 11], Gupta and Shandil [12]). For a detailed study of Marangoni convection, one may be referred to the works of Koschmieder [13], Normand et al. [14] and Schatz et al. [15].

Linear stability analysis of Pearson 6] with the no-slip boundary condition replaced by the free-slip boundary condition for the conducting case of the lower boundary has been analysed by Boeck and Thess [16] and obtained the critical Marangoni number M_c = 57.598 and the critical wave number a_c = 1.7003. However, they did not consider for the insulating case of the lower boundary. In this investigation, therefore, we have considered the problem of the onset of Marangoni convection in a horizontal fluid layer heated from below for the insulating case of the lower boundary with free-slip condition and obtained that the critical Marangoni number M_c = 24 and the critical wave number a_c tends to 0 when the Biot number L = 0. Further that the critical Marangoni number becomes asymptotically proportional to the the Biot number at the upper surface for large values of the later while the critical wave number remains finite. The nature of the neutral state is examined by solving the characteristic value equation for the underlying problem numerically. It is shown that the neutral state is stationary rather than an oscillatory one.

2.1 Formulation of the Problem

We consider an infinite horizontal layer of viscous fluid of uniform thickness d at rest, whose lower surface is assumed to be free and insulating, and the upper one to be free where surface tension gradients arise due to temperature perturbations. We choose a Cartesian coordinate system of axes with the x and y axes in the plane of the lower surface and the z axis along the vertically upward direction so that the fluid is confined between the planes at $z=0$ and $z=d$. A temperature gradient is maintained across the layer by maintaining the lower boundary at a constant temperature $T_0(>0)$ and the upper boundary at $T_1<(T_0)$. It is assumed that surface tension is given by the simple linear law $\tau=\tau_1-\sigma(T-T_1)$ where the constant τ_1 is the unperturbed value of τ at the unperturbed surface temperature $T=T_1$ and $-\sigma=(\partial\tau/\partial T)_{T/T_1}$ represents the rate of change of surface tension with temperature, evaluated at temperature T_1, and surface tension being a monotonically decreasing function of temperature, σ is positive.

The linearized equations governing the small perturbations in the relevant context (neglecting buoyancy) are given as

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2 w\right)\nabla^2 w=0 \tag{1}$$

$$\left(\frac{\partial\theta}{\partial t}-\beta w\right)=\kappa\nabla^2\theta \tag{2}$$

where w and θ denote respectively the z-component of velocity perturbation and temperature perturbation; β is the temperature gradient which is maintained; ν is the kinematic viscosity; κ is the thermal diffusivity and are each assumed constant. $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ and t denotes time.

In seeking solutions of the equations (1) and (2), we must satisfy certain boundary conditions, The boundary conditions at the lower free and insulating surface z = 0 are straightforward and given by

$$w = 0 \tag{3}$$

$$\frac{\partial^2 w}{\partial z^2} = 0 \tag{4}$$

$$\frac{\partial \theta}{\partial z} = 0 \tag{5}$$

The boundary conditions at the upper free surface $z = d$ are more complicated.
Because of the non-deflecting surface, the normal component of the velocity must vanish, that is,

$$w = 0 \tag{6}$$

The stress-balance condition satisfy the equation

$$\rho \nu \frac{\partial^2 w}{\partial z^2} = \sigma \nabla_1^2 \theta \tag{7}$$

here ρ is the density and $\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The boundary condition (7) is usually referred to as the Marangoni boundary condition (Pearson [6]). Finally, if we consider conservation of heat transport across the upper free surface, then we have

$$-k \frac{\partial \theta}{\partial z} = q\theta \tag{8}$$

where k is the thermal conductivity of the fluid and q is the heat transfer coefficient.
We now suppose that the perturbations w and θ are of the form

$$w(x, y, z, t) = w(z) \exp[i(a_x x + a_y y) + pt]$$

$$\theta(x, y, z, t) = \theta(z) \exp[i(a_x x + a_y y) + pt]$$

where $a = \sqrt{a_x^2 + a_y^2}$ is the wave number of the disturbance and p is a time constant (which can be complex). We now introduce the non-dimensional quantities using $d, \nu/d, d^2/\nu$ and $\beta d\nu/\kappa$ as the appropriate scales for length, velocity, time and temperature respectively and putting $z_* = z/d$, $a_* = ad$, $p_* = pd^2/\nu$, $w_* = wd/\nu$, $\theta_* = \theta\kappa/\beta d\nu$, and $D_* = d(d/dz)$. We now let x, y and z stand for co-ordinates in the new units and omitting asterisk for simplicity, equations (1)-(2) and boundary conditions (3)-(8) can be reduced to the following non-dimensional form

$$(D^2 - a^2)(D^2 - a^2 - p)w = 0 \tag{9}$$

$$\left(D^2 - a^2 - pP_r\right)\theta = -w \tag{10}$$

$$\left.\begin{aligned} w(0) &= 0 \\ D^2 w(0) &= 0 \\ D\theta(0) &= 0 \end{aligned}\right\} \tag{11a, b, c}$$

evaluated on the lower free boundary $z = 0$, and

$$\left.\begin{aligned} w(1) &= 0 \\ D^2 w(1) &= -a^2 M \theta(1) \\ D\theta(1) &= -L\theta(1) \end{aligned}\right\} \tag{12a, b, c}$$

evaluated on the upper free surface $z = 1$, where $M = \frac{\sigma \beta d^2}{\rho \kappa \nu}$ is the Marangoni number, $P_r = \frac{\nu}{\kappa}$ is the Prandtl number and $L = \frac{qd}{k}$ is the Biot number.

We now consider the case when instability sets in as stationary convection, that is, the marginal state is, therefore, characterized by $p = 0$ and equations (8) and (9) become

$$(D^2 - a^2)^2 w = 0 \tag{13}$$

$$(D^2 - a^2)\theta = -w \tag{14}$$

Solution to equations (13)–(14) is sought subject to boundary conditions (11a, b, c)–(12a, b, c). Thus we have an eigenvalue problem.

3.1 Solution of the Problem

The solution of equation (13) subject to boundary conditions (11a, b) and (12a) is given by

$$w = A[zC_{az} - \frac{S_{az}}{tanh\, a}] \tag{15}$$

where $S_{az} = \sinh az$, $C_{az} = \cosh az$ and A is an arbitrary constant.

The solution of equation (14), using the expression (15) for *w*, and boundary conditions (11c) and (12c) is given by

$$\theta = -A[\{-\frac{1}{4a} z^2 S_{az} + \frac{S_a + 2aC_a}{4a^2 S_a} zC_{az} - \frac{S_a + 2aC_a}{4a^3 S_a} S_{az}\}$$
$$+ \{\frac{a^2 S_a (S_a - aC_a) + L(S_a^2 + aS_a C_a - a^2(2 + S_a^2))}{4a^3 S_a (aS_a + LC_a)} C_{az}\}] \tag{16}$$

where $S_a = \sinh a$ and $C_a = \cosh a$. Now substitution from equations (15) and (16) into the remaining boundary condition (12b) yields finally the neutral stability condition

$$M = \frac{8aS_a (aS_a + LC_a)}{(S_a^2 - a^2)} \tag{17}$$

4.1 Numerical Results and Discussion

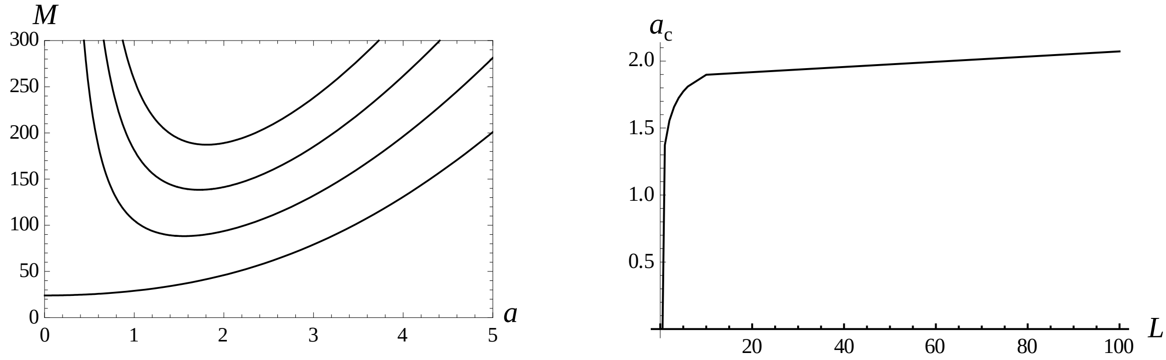
Since some of the algebraic manipulations involved are rather lengthy a symbolic algebra package is used to compute the minimum values of Marangoni number M with respect to wave number a for fixed values of the Biot number L, using relation (17). The critical Marangoni number M_c and the corresponding wave number a_c for various values of L are presented in Table 1.

Table 1. The numerical values of M_c and a_c for various values of L.

L	M_c	a_c
0	24	0.0003
2	88.258	1.5564
4	138.408	1.7247
5	162.923	1.7729
10	283.735	1.8979
100	2428.16	2.0722
1000	23848.8	2.0956
10^{10}	2.3800×10^{10}	2.0983

It is found that, as L increases, the critical Marangoni number M_c as well as the corresponding wave number a_c increase. In the limit $L \to \infty$ we find that M_c becomes asymptotically proportional to L while $a_c \to a_{max} \approx 2.0983$ remains finite. In particular, for $L = 0$ it is interesting to note that $M_c = 24$, $a_c \to 0$ to be compared with $M_c = 48$, $a_c = 0$ as obtained by Pearson [6] for the corresponding case with no-slip boundary condition at the lower boundary. The higher critical Marangoni number in the no-slip case are due to the stabilizing effect of friction at the bottom, which is also known from the buoyancy driven convection (Chandrasekhar [4]).

The relation (17) is plotted in Fig.1(a), as the neutral stability curves for L = 0, 2, 4 and 6 For given value of L, the instability threshold is given by the minimum of M with respect to a. The wave number a_c for various values of L is plotted in Fig. 1(b).



(a) (b)

Fig. 1(a) Neutral stability curves for L = 0, 2, 4, 6. The mimima of these curves represent the instability threshold for given L. (b) Wave number a_c as a function of L. In the limit $L \to \infty$, a_c approaches the finite value 2.0983.

5.1 Nature of the Neutral State

Owing to the peculiar nature of the stress-balance condition, it is difficult to prove analytically the validity of the 'principle of exchange of stabilities' for the present problem. Since the assumption of this principle is a key step in the analysis, it is desirable to settle the question numerically. For the Pearson's problems Vidal and Acrivos [17] have established that the neutral state is stationary rather than an oscillatory one, numerically. The method of numerical approach used here is similar to that of Vidal and Acrivos; the exact solution to the differential equations is substituted into the boundary conditions, leaving a complex number for the eigen value, the Marangoni number. Since this number is real (a ratio of physical parameters), solution can exist only if the imaginary part of the complex number is zero. The parameter space is searched numerically for situations where this is true.

To keep the notation the same as that of Vidal and Acrivos, the change of variable $p_1 = pP_r$ is made. Then equations (9) and (10) may be written as

$$\left.\begin{aligned} [p_1 - P_r(D^2-a^2)](D^2-a^2)w &= 0 \\ \text{and} \qquad [D^2-a^2-p_1]\theta &= -w \end{aligned}\right\} \qquad (18)$$

In order to reduce the amount of numerical work, these equations will be solved subject to the following simpler but still representative boundary conditions:

$$\left.\begin{aligned} w(0) = D^2w(0) = w(1) &= 0 \\ \theta(0) = D\theta(1) &= 0 \\ D^2w(1) + Ba^2\theta(1) &= 0 \end{aligned}\right\} \qquad (19)$$

For a non-stationary neutral state $p_1 = iS$ with S real, in which case the solution of equations (18) and (19) may be solved analytically to obtain the relationship

$$a_3 + ia_4 = M(a_1 + ia_2) \qquad (20)$$

where a_1, a_2, a_3, a_4 are real valued analytic functions of a, S and P_r.

Solving equation (20) for M, we obtain

$$M = \frac{a_1a_3 + a_2a_4}{a_1^2 + a_2^2} + i\frac{a_1a_4 - a_2a_3}{a_1^2 + a_2^2} \tag{21}$$

Since M must clearly be real, $$a_1a_4 - a_2a_3 = 0 \tag{22}$$

The function $a_1a_4 - a_2a_3$ was computed for P_r = 0.1, 0.5, 0.9, 5.0, 10^2 and 10^3, a was varied from 0.1 to 10 in steps of 0.1 and S was varied from 0 to 10 in steps 0.1. In all cases, the only real root of $a_1a_4 - a_2a_3 = 0$ was found to occur at $S = 0$. This indicates that the marginal or neutral state is stationary.

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