INTERNATIONAL JOURNAL OF ADVANCED SCIENTIFIC AND TECHNICAL RESEARCH
(ISSUE 2, VOLUME 4- August 2012) ISSN 2249-9954

A NOTE ON SOME GROWTH PROPERTIES OF WRONSKIANS BY MEANS OF L*-ORDER

[1]SANJIB KUMAR DATTA, [2]BIBHAS CHANDRA GIRI AND [3]SANTONU SAVAPONDIT
[1]Department of Mathematics, University of Kalyani,
Kalyani, Dist.-Nadia, Pin-741235,
West Bengal, India.

(Former Address: Department of Mathematics, University of North Bengal, Raja Rammohunpur, Dist.-Darjeeling, Pin-734013, West Bengal, India.)
[2]Department of Mathematics, Jadavpur University,
Kolkata, Pin-700032, West Bengal, India.

[3]Department of Mathematics,
Sikkim Manipal Institute of Technology,
Majitar, Pin - 737136, Sikkim, India.

Abstract

The aim of this paper is to study the comparative growth properties of composite entire or meromorphic functions and wronskians generated by one of the factors using L*-order .

AMS Subject Classification: 30D35, 30D30

Key words and phrases: Entire function, meromorphic function, wronskian, growth property, L*-order.

1 Introduction, Definitions and Notations.

Somasundaram and Thamizharasi [7] introduced the notion of L-order and L-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a.

The L*-order and the L*-type of a meromorphic function are the more generalised concepts of L-order and L-type respectively. The following definitions are well known.

Definition 1 *The L^*-order $\rho_f^{L^*}$ and L^*-lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined as*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}.$$

If f is entire, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}$$

where

$$\log^{[k]} x = \log(\log^{[k-1]} x) \text{ for } k = 1,2,3,\ldots \text{ and } \log^{[0]} x = x.$$

Definition 2 *The hyper L^*-order $\overline{\rho}_f^{L^*}$ and hyper L^*-lower order $\overline{\lambda}_f^{L^*}$ of a meromorphic function f are defined as*

$$\overline{\rho}_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \overline{\lambda}_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log\left[re^{L(r)}\right]}.$$

If f is entire then

$$\overline{\rho}_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \overline{\lambda}_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log\left[re^{L(r)}\right]}.$$

Definition 3 *[6]Let f be a meromorphic function of L^*-order zero. Then the quantities ρ_f^*, λ_f^* and $\overline{\rho}_f^*$, $\overline{\lambda}_f^*$ are defined in the following way*

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r} \ , \ \lambda_f^* = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r}$$

$$\text{and } \overline{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r} \ , \ \overline{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}.$$

If f is entire then clearly

$$\begin{aligned} \rho_f^* &= \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r} \ , \ \lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r} \ , \\ \text{and } \overline{\rho}_f^* &= \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r} \ , \ \overline{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}. \end{aligned}$$

Definition 4 *The L^*-type $\sigma_f^{L^*}$ of a meromorphic function f is defined as*

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

When f is entire then

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}, \quad 0 < \rho_f^{L^*} < \infty.$$

Definition 5 *A meromorphic function $a \equiv a(z)$ is called small with respect to f if*

$$T(r,a) = S(r,f).$$

Definition 6 *Let $a_1, a_2, ...a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $W(f) = W(a_1, a_2, ...a_k; f)$ the Wronskian determinant of $a_1, a_2, ...a_k, f$. i.e.,*

$$W(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1' & a_2' & . & . & . & a_k' & f' \\ . & & & & & & . \\ . & & & & & & . \\ . & & & & & & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

Definition 7 *If $a \in \mathbb{C} \cup \{\infty\}$ the quantity*

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a,f)}{T(r,f)} = \liminf_{r\to\infty} \frac{m(r,a,f)}{T(r,f)}$$

is called the Nevanlinna's deficiency of the value of 'a'.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a,f) > 0$ is countable and $\sum\limits_{a\neq\infty} \delta(a,f) + \delta(\infty,f) \leq 2$. If in particular $\sum\limits_{a\neq\infty} \delta(a,f) + \delta(\infty,f) = 2$, we say that f has the maximum deficiency sum.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 *[4]If f and g be two entire functions then for all sufficiently large values of r,*

$$M(r, fog) \geq M\left(\frac{1}{8} M\left(\frac{r}{2}, g\right) - |g(0)|, f\right).$$

Lemma 2 *[1]Let f be meromorphic and g be entire then for all suffiently large values of r,*

$$T(r, fog) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T\left(M(r,g), f\right).$$

Lemma 3 *[3]Let f be meromorphic and g be entire and suppose that $0 < \mu \leq \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity,*

$$T(r, fog) \geq T\left(\exp\left(r^{\mu}\right), f\right).$$

Lemma 4 *[5]Let f be a transcendental meromorphic function having the maximum deficiency sum. Then*

$$\lim_{r\to\infty} \frac{T(r, W(f))}{T(r,f)} = 1 + k - k\delta\left(\infty, f\right).$$

Lemma 5 *Let f be a transcendental meromorphic function with the maximum deficiency sum then the L^*-order and L^*-lower order of $W(f)$ are same as those of f and the L^*-type of $W(f)$ is $\{1 + k - k\delta\left(\infty, f\right)\}$ times that of f.*

Proof. By Lemma 4

$$\lim_{r\to\infty} \frac{T(r, W(f))}{T(r,f)} \text{exists and is equal to 1.}$$

So
$$\begin{aligned} \rho_{W(f)}^{L^*} &= \limsup_{r\to\infty} \frac{\log T(r, W(f))}{\log\left[re^{L(r)}\right]} \\ &= \limsup_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]} . \lim_{r\to\infty} \frac{\log T(r, W(f))}{\log T(r,f)} \\ &= \rho_f^{L^*}.1 = \rho_f^{L^*}. \end{aligned}$$

Also,
$$\begin{aligned} \lambda_{W(f)}^{L^*} &= \liminf_{r\to\infty} \frac{\log T(r, W(f))}{\log\left[re^{L(r)}\right]} \\ &= \liminf_{r\to\infty} \frac{\log T(r,f)}{\log\left[re^{L(r)}\right]} . \lim_{r\to\infty} \frac{\log T(r, W(f))}{\log T(r,f)} \\ &= \lambda_f^{L^*}.1 = \lambda_f^{L^*}. \end{aligned}$$

Further,
$$\begin{aligned} \sigma_{W(f)}^{L^*} &= \limsup_{r\to\infty} \frac{T(r, W(f))}{\left[re^{L(r)}\right]^{\rho_{W(f)}^{L^*}}} \\ &= \lim_{r\to\infty} \frac{T(r, W(f))}{T(r,f)} . \limsup_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_{W(f)}^{L^*}}} \\ &= \left\{1 + k - k\delta\left(\infty, f\right)\right\} . \limsup_{r\to\infty} \frac{T(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \\ &= \left\{1 + k - k\delta\left(\infty, f\right)\right\} . \sigma_f^{L^*}. \end{aligned}$$

This proves the lemma. ■

Lemma 6 *Let f be a transcendental meromorphic function having the maximum deficiency sum then the L*-hyper order (L*-hyper lower order) of $W(f)$ and f are equal.*

The proof of Lemma 6 is omitted as it can be carried out in the line of Lemma 5.

Lemma 7 *Let f be meromorphic and g be transcendental entire such that*

$$\rho_f = 0 \text{ and } \rho_g^{L^*} < \infty \text{ then } \rho_{fog}^{L^*} \leq \rho_f^* . \rho_g^{L^*}.$$

Proof. *In view of Lemma 2 and the inequality*

$$T(r,g) \leq \log^+ M(r,g)$$

we get that

$$\begin{aligned}
\rho_{fog}^{L^*} &= \limsup_{r\to\infty} \frac{\log T(r,fog)}{\log\left[re^{L(r)}\right]} \\
&\leq \limsup_{r\to\infty} \frac{\log T(M(r,g),f)+o(1)}{\log\left[re^{L(r)}\right]} \\
&\leq \limsup_{r\to\infty} \frac{\log T(M(r,g),f)}{\log^{[2]} M(r,g)} \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log\left[re^{L(r)}\right]} \\
&= \rho_f^* \rho_g^{L^*}.
\end{aligned}$$

This proves the lemma. ■

3 Theorems.

In this section we present the main results of the paper.

Theorem 1 *Let f be transcendental meromorphic and g be entire satisfying the following conditions $(i)\rho_f^{L^*}$ and $\rho_g^{L^*}$ are both finite, $(ii)\rho_f^{L^*}$ is positive and $(iii) \sum\limits_{a\neq\infty} \delta(a,f) + \delta(\infty,f) = 2.$ Then for each $\alpha \in (-\infty,\infty)$,*

$$\liminf_{r\to\infty} \frac{\{\log T(r,fog)\}^{1+\alpha}}{\log T\left\{\exp(r^{p'}),W(f)\right\}} = 0 \text{ if } p' > (1+\alpha)\rho_g^{L^*}.$$

Proof. If $1+\alpha \leq 0$. Then the theorem is trivial. So we take $1+\alpha > 0.$ Since $T(r,g) \leq \log^+ M(r,g)$ by Lemma 2 we get for all sufficiently large values of r,

$$T(r,fog) \leq \{1+o(1)\} T\left\{M(r,g),f\right\}$$

$$\text{i.e.} \qquad \log T(r,fog) \leq \log\{1+o(1)\} + \log T(M(r,g),f)$$

$$\begin{aligned}
\text{i.e.,} \quad & \log T(r, fog) \\
\leq \; & o(1) + (\rho_f^{L^*} + \epsilon) \log \left\{ M(r,g) e^{L(M(r,g))} \right\} \\
= \; & o(1) + (\rho_f^{L^*} + \epsilon) \{ \log M(r,g) + L(M(r,g)) \} \\
\leq \; & o(1) + (\rho_f^{L^*} + \epsilon) \left[r e^{L(r)} \right]^{(\rho_{g.}^{L^*} + \epsilon)} + (\rho_f^{L^*} + \epsilon) L(M(r,g)) \\
= \; & \left[r e^{L(r)} \right]^{(\rho_{g.}^{L^*} + \epsilon)} \{ (\rho_f^{L^*} + \epsilon) + o(1) \} + (\rho_f^{L^*} + \epsilon) L(M(r,g))
\end{aligned}$$

$$\text{i.e.,} \quad \{\log T(r, fog)\}^{1+\alpha}$$

$$\leq \left[\left\{ r e^{L(r)} \right\}^{(\rho_{g.}^{L^*} + \epsilon)} \{ (\rho_f^{L^*} + \epsilon) + o(1) \} + (\rho_f^{L^*} + \epsilon) L(M(r,g)) \right]^{1+\alpha} . \quad (1)$$

Again we have for a sequence of r tending to infinity and for $\epsilon\,(> 0)$,

$$\begin{aligned}
\log T \left\{ \exp(r^{p'}), W(f) \right\} \geq \; & (\rho_{W(f)}^{L^*} - \epsilon) \log \left[\exp(r^{p'}) \exp \left\{ L \left(\exp \left(r^{p'} \right) \right) \right\} \right] \\
= \; & (\rho_f^{L^*} - \epsilon) \left[r^{p'} + L \left(\exp \left(r^{p'} \right) \right) \right] . \quad (2)
\end{aligned}$$

So from (1) and (2) we get that

$$\begin{aligned}
& \frac{\{\log T(r, fog)\}^{1+\alpha}}{\log T \{\exp(r^{p'}), W(f)\}} \\
\leq \; & \frac{\left[\left\{ r e^{L(r)} \right\}^{(\rho_{g.}^{L^*} + \epsilon)} (\rho_f^{L^*} + \epsilon + o(1)) + (\rho_f^{L^*} + \epsilon) L(M(r,g)) \right]^{1+\alpha}}{(\rho_f^{L^*} - \epsilon)[r^{p'} + L \{\exp(r^{p'})\}]} .
\end{aligned}$$

Let

$$\left\{ e^{L(r)} \right\}^{(\rho_{g.}^{L^*} + \epsilon)} \left\{ \rho_f^{L^*} + \epsilon + o(1) \right\} = k_1, \; (\rho_f^{L^*} + \epsilon) L(M(r,g)) = k_2,$$

$$\rho_f^{L^*} - \epsilon = k_3 \text{ and } (\rho_f^{L^*} - \epsilon) L \left(\exp \left(r^{p'} \right) \right) = k_4 .$$

$$\begin{aligned}
\text{Then} \quad \frac{\{\log T(r, fog)\}^{1+\alpha}}{\log T \{\exp(r^{p'}), W(f)\}} \leq \; & \frac{\{ r^{(\rho_g^{L^*} + \epsilon)} k_1 + k_2 \}^{1+\alpha}}{k_3 r^{p'} + k_4} \\
= \; & \frac{r^{(\rho_{g.}^{L^*} + \epsilon)(1+\alpha)} \left\{ k_1 + \frac{k_2}{r^{(\rho_g^{L^*} + \epsilon)}} \right\}^{1+\alpha}}{k_3 r^{p'} + k_4}
\end{aligned}$$

where k_1, k_2, k_3 and k_4 are finite.

$$\text{Since} \qquad (\rho_g^{L^*} + \epsilon)(1 + \alpha) < p'$$

$$\text{therefore} \qquad \liminf_{r\to\infty} \frac{\{\log T(r, fog)\}^{1+\alpha}}{\log T\{\exp(r^{p'}), W(f)\}} = 0$$

where we choose $\epsilon\,(>0)$ such that

$$0 < \epsilon < \min\left\{\rho_f^{L^*}, \frac{p'}{1+\alpha} - \rho_g^{L^*}\right\}.$$

which proves the theorem. ■

Theorem 2 *If f be meromorphic and g be transcendental entire such that $\rho_{g.}^{L^*} < \infty, \rho_{fog}^{L^*} = \infty$ and $\sum\limits_{a\neq\infty} \delta(a, g) + \delta(\infty, g) = 2$. Then for every $A > 0$*

$$\limsup_{r\to\infty} \frac{\log T(r, fog)}{\log T(r^A, W(g))} = \infty.$$

Proof. If possible let there exists a constant β such that for all sufficiently large values of r we have

$$\log T(r, fog) \le \beta \log T(r^A, W(g)). \tag{3}$$

In view of Lemma 5 for all sufficiently large values of r we get that

$$\begin{aligned} \log T(r^A, W(g)) &\le (\rho_{W(g)}^{L^*} + \epsilon) \log\left[r^A \exp\left\{L(r^A)\right\}\right] \\ \text{i.e. } \log T(r^A, W(g)) &\le (\rho_g^{L^*} + \epsilon)\left\{A \log r + L(r^A)\right\}. \end{aligned} \tag{4}$$

Now combining (3) and(4) we obtain for all sufficiently large values of r

$$\log T(r, fog) \le \beta(\rho_g^{L^*} + \epsilon)\left\{A \log r + L(r^A)\right\}$$

$$\begin{aligned} \text{which implies that} \qquad \frac{\log T(r, fog)}{\log\left[re^{L(r)}\right]} &\le \frac{\beta(\rho_g^{L^*} + \epsilon)\left\{A \log r + L(r^A)\right\}}{\log\left[re^{L(r)}\right]} \\ &= \beta(\rho_g^{L^*} + \epsilon)\frac{\left\{A \log r + L(r^A)\right\}}{\log\left[re^{L(r)}\right]}. \end{aligned}$$

$$\text{Therefore} \qquad \frac{\log T(r, fog)}{\log\left[re^{L(r)}\right]} \le \beta.A.(\rho_g^{L^*} + \epsilon)$$

$$\text{i.e,} \qquad \rho_{fog}^{L^*} \le \beta.A.(\rho_g^{L^*} + \epsilon),$$

which contradicts the condition $\rho_{fog}^{L^*} = \infty$. So for a sequence of values of r tending to infinity, it follows that $\log T(r, fog) > \beta \log T(r^A, W(g))$ from which the theorem follows. ■

Corollary 1 *Under the assumption of Theorem 2*

$$\limsup_{r\to\infty} \frac{T(r, fog)}{T(r^A, W(g))} = \infty.$$

Proof. By Theorem 2 we obtain for all sufficiently large values of r and for $K > 1$,

$$\begin{aligned} \log T(r, fog) &> K \log T(r^A, W(g)) \\ \text{i.e.} \quad T(r, fog) &> \{T(r^A, W(g))\}^K \end{aligned}$$

from which the corollary follows. ■

Remark 1 *If we take* $\rho_f^{L^*} < \infty$ *and* $\sum_{a\neq\infty} \delta(a, f) + \delta(\infty, f) = 2$ *instead of* $\rho_g^{L^*} < \infty$ *and* $\sum_{a\neq\infty} \delta(a, g) + \delta(\infty, g) = 2$ *respectively then Theorem 2 and Corollary 1 remains valid with* $W(g)$ *replaced by* $W(f)$ *in the denominator.*

Theorem 3 *Let* f *and* g *be two entire functions with* $\lambda_f^{L^*} > 0$ *and* $\rho_f^{L^*} < \lambda_g^{L^*}$*. Also let* f *be transcendental with* $\sum_{a\neq\infty} \delta(a, f) + \delta(\infty, f) = 2$*. Then*

$$\lim_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log M(r, W(f))} = \infty.$$

Proof. In view of Lemma 1, we have for all sufficiently large values of r,

$$M(r, fog) \geq M\left(\frac{1}{16} M\left(\frac{r}{2}, g\right), f\right)$$

$$\begin{aligned} \text{i.e.} \quad \log^{[2]} M(r, fog) &\geq \log^{[2]} M\left(\frac{1}{16} M\left(\frac{r}{2}, g\right), f\right) \\ \text{i.e.} \quad \log^{[2]} M(r, fog) &\geq \left(\lambda_f^{L^*} - \epsilon\right) \log\left(\frac{1}{16} M\left(\frac{r}{2}, g\right) e^{L\left(\frac{1}{16} M\left(\frac{r}{2}, g\right)\right)}\right) \\ \text{i.e.} \quad \log^{[2]} M(r, fog) &\geq \left(\lambda_f^{L^*} - \epsilon\right) \log\frac{1}{16} + \left(\lambda_f^{L^*} - \epsilon\right) \log M\left(\frac{r}{2}, g\right) \\ &\quad + \left(\lambda_f^{L^*} - \epsilon\right) L\left(\frac{1}{16} M\left(\frac{r}{2}, g\right)\right) \\ \text{i.e.} \quad \log^{[2]} M(r, fog) &\geq O(1) + \left(\lambda_f^{L^*} - \epsilon\right) \left(\frac{r}{2} e^{L\left(\frac{r}{2}\right)}\right)^{\lambda_g^{L^*} - \epsilon} \\ &\quad + \left(\lambda_f^{L^*} - \epsilon\right) L\left(\frac{1}{16} M\left(\frac{r}{2}, g\right)\right). \end{aligned} \tag{5}$$

Again for all sufficiently large values of r we get by Lemma 5 that

$$\log M(r, W(f)) \leq \left(re^{L(r)}\right)^{\rho_{W(f)}^{L^*} + \epsilon} = \left(re^{L(r)}\right)^{\rho_f^{L^*} + \epsilon}. \tag{6}$$

Now combining (5) and (6) it follows from all sufficiently large values of r,

$$\begin{aligned}
&\frac{\log^{[2]} M(r, fog)}{\log M(r, W(f))} \\
\geq\ & \frac{O(1) + \left(\lambda_f^{L^*} - \epsilon\right)\left[\frac{r}{2}e^{L\left(\frac{r}{2}\right)}\right]^{\lambda_g^{L^*}-\epsilon} + \left(\lambda_f^{L^*} - \epsilon\right) L\left(\frac{1}{16}M\left(\frac{r}{2}, g\right)\right)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}+\epsilon}}. \quad (7)
\end{aligned}$$

Since $\rho_f^{L^*} < \lambda_g^{L^*}$ we can choose $\epsilon\,(> 0)$ in such a way that

$$\rho_f^{L^*} + \epsilon < \lambda_g^{L^*} - \epsilon. \quad (8)$$

Thus from (7) and (8) we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, fog)}{\log M(r, W(f))} = \infty,$$

from which the theorem follows. ■

Theorem 4 *If f be a transcendental meromorphic function and g be entire with $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty, \rho_g^{L^*} < \infty$ and $\sum_{a\neq\infty} \delta(a, f) + \delta(\infty, f) = 2$. Then*

$$\lim_{r\to\infty} \frac{T(r, fog)T(r, W(f))}{T\left[\exp\left(r^{p'}\right), W(f)\right]} = 0 \text{ if } p' > \rho_g^{L^*}.$$

Proof. Since $T(r, g) \leq \log^+ M(r, g)$, for all sufficiently large values of r we get from Lemma 2

$$T(r, fog) \leq \{1 + o(1)\}\, T(M(r, g), f)$$

$$\begin{aligned}
\text{i.e.,}\quad & T(r, fog) \\
\leq\ & \{1 + o(1)\} \exp\left(\left(\rho_f^{L^*} + \epsilon\right)\left(\left(re^{L(r)}\right)^{\rho_g^{L^*}+\epsilon} + L(M(r, g))\right)\right) \\
=\ & \{1 + o(1)\} \exp\left(\left(\rho_f^{L^*} + \epsilon\right)\left(re^{L(r)}\right)^{\rho_g^{L^*}+\epsilon}\right) \\
& .\exp\left(\left(\rho_f^{L^*} + \epsilon\right) L(M(r, g))\right). \quad (9)
\end{aligned}$$

Again by Lemma 5 for all sufficiently large values of r,

$$T(r, W(f)) \leq \left(re^{L(r)}\right)^{\rho_{W(f)}^{L^*}+\epsilon} = \left(re^{L(r)}\right)^{\rho_f^{L^*}+\epsilon}. \quad (10)$$

Now combining (9) and (10) it follows for all sufficiently large values of r,

$$\begin{aligned} & T(r, fog)T(r, W(f)) \\ \leq \; & \{1+o(1)\} \exp\left(\left(\rho_f^{L^*}+\epsilon\right)\left(re^{L(r)}\right)^{\rho_g^{L^*}+\epsilon}\right) \\ & .\exp\left(\left(\rho_f^{L^*}+\epsilon\right) M(r,g)\right)\left(re^{L(r)}\right)^{\rho_f^{L^*}+\epsilon}. \end{aligned} \tag{11}$$

Also in view of Lemma 5, we have for all sufficiently large values of r that

$$\log T\left[\exp\left(r^{p'}\right), W(f)\right] \geq \left(\lambda_{W(f)}^{L^*}-\epsilon\right)\log\left[\exp\left(r^{p'}\right)\exp\left\{L(\exp\left(r^{p'}\right))\right\}\right]$$

$$\begin{aligned} & \text{i.e.,} \quad T\left\{\exp\left(r^{p'}\right), W(f)\right\} \\ \geq \; & \left[\exp\left(r^{p'}\right)\exp\left\{L\left(\exp\left(r^{p'}\right)\right)\right\}\right]^{\lambda_{W(f)}^{L^*}-\epsilon} \\ = \; & \exp\left[\left(\lambda_{W(f)}^{L^*}-\epsilon\right)r^{p'}\right]\left[\exp\left\{L\left(\exp\left(r^{p'}\right)\right)\right\}\right]^{\lambda_{W(f)}^{L^*}-\epsilon} \\ = \; & \exp\left[\left(\lambda_f^{L^*}-\epsilon\right)r^{p'}\right]\left[\exp\left\{L\left(\exp\left(r^{p'}\right)\right)\right\}\right]^{\lambda_{W(f)}^{L^*}-\epsilon}. \end{aligned} \tag{12}$$

From (11) and (12) it follows for all sufficiently large values of r,

$$\begin{aligned} & \frac{T(r, fog)T(r, W(f))}{T\left\{\exp\left(r^{p'}\right), W(f)\right\}} \\ \leq \; & \frac{\exp\left[\left(\rho_f^{L^*}+\epsilon\right)\left(re^{L(r)}\right)^{\rho_g^{L^*}+\epsilon}\right]\exp\left[\left(\rho_f^{L^*}+\epsilon\right)M(r,g)\right]}{\exp\left(\left(\lambda_f^{L^*}-\epsilon\right)r^{p'}\right)\left[\exp\left\{L\left(\exp\left(r^{p'}\right)\right)\right\}\right]^{\lambda_{W(f)}^{L^*}-\epsilon}} \\ & .\{1+o(1)\}\left(re^{L(r)}\right)^{\rho_f^{L^*}+\epsilon}. \end{aligned} \tag{13}$$

As $p' > \rho_g^{L^*}$, so we can choose $\epsilon\,(>0)$ such that

$$p' > \rho_g^{L^*}+\epsilon. \tag{14}$$

Thus the theorem follows from (13) and (14). ■

Theorem 5 *Let f be a transcendental meromorphic function and g be a transcendental entire function such that $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ and $\sum_{a\neq\infty}\delta(a,f)+\delta(\infty,f)=2$. Then for every $A>0$*

$$\lim_{r\to\infty}\frac{\log T(r, fog)}{\log T(r^A, W(f))} = \infty.$$

If further $\rho_g^{L^*} < \infty$ *and* $\sum_{a \neq \infty} \delta(a, g) + \delta(\infty, g) = 2$ *then*

$$\lim_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, W(g))} = \infty.$$

Proof. Since $\lambda_f^{L^*} > 0$, $\lambda_{fog}^{L^*} = \infty \{cf.[2]\}$. So it follows that for arbitrary large N and for all sufficiently large values of r,

$$\log T(r, fog) > AN \log \left[re^{L(r)} \right]. \tag{15}$$

Again since $\rho_f^{L^*} < \infty$, for all sufficiently large values of r we get by Lemma 5,

$$\log T(r^A, W(f)) < \left(\rho_f^{L^*} + 1 \right) \log \left[r^A e^{L(r^A)} \right]. \tag{16}$$

Again now from (15) and (16) it follows for all sufficiently large values of r that

$$\frac{\log T(r, fog)}{\log T(r^A, W(f))} > \frac{AN \log \left[re^{L(r)} \right]}{\left(\rho_f^{L^*} + 1 \right) \log \left[r^A e^{L(r^A)} \right]}.$$

$$\text{Hence} \qquad \frac{\log T(r, fog)}{\log T(r^A, W(f))} > \frac{AN \left[\log r + L(r) \right]}{\left(\rho_f^{L^*} + 1 \right) \left[A \log r + L(r^A) \right]}$$

$$\text{and so} \quad \lim_{r \to \infty} \frac{\log T(r, fog)}{\log T(r^A, W(f))} = \infty.$$

Again since $\rho_g^{L^*} < \infty$, for all sufficiently large values of r we get by Lemma 5,

$$\begin{aligned} \log T(r^A, W(g)) &< \left(\rho_g^{L^*} + 1 \right) \log \left[r^A e^{L(r^A)} \right] \\ &= \left(\rho_g^{L^*} + 1 \right) \left[A \log r + L(r^A) \right]. \end{aligned} \tag{17}$$

Now from (15) and (17) it follows for all sufficiently large values of r that

$$\begin{aligned} \frac{\log T(r, fog)}{\log T(r^A, W(g))} &> \frac{AN \log \left[re^{L(r)} \right]}{\left(\rho_g^{L^*} + 1 \right) \left[A \log r + L(r^A) \right]} \\ &= \frac{AN \left[\log r + L(r) \right]}{\left(\rho_g^{L^*} + 1 \right) \left[A \log r + L(r^A) \right]}. \end{aligned} \tag{18}$$

Thus the theorem follows from (18). ■

Theorem 6 *Let* f *be a transcendental meromorphic function with* $0 < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$ *and* $\sum_{a \neq \infty} \delta(a, f) + \delta(\infty, f) = 2$ *and* g *be entire. Then*

$$\limsup_{r \to \infty} \frac{\log^{[2]} T \left\{ \exp \left(r^{\rho_g^{L^*}} \right), fog \right\}}{\log T(\exp(r^\mu), W(f))} = \infty \text{ where } 0 < \mu < \rho_g^{L^*}.$$

Proof. Let $0 < \mu' < \rho_g^{L^*}$. Then in view of Lemma 3 we get for a sequence of values of r tending to infinity,

$$\log T(r, fog) \geq \log T\left(\exp\left(r^{\mu'}\right), f\right)$$

$$\begin{aligned}
\text{i.e.,}\quad & \log T(r, fog) \geq (\lambda_f^{L^*} - \epsilon) \log\left[\exp\left(r^{\mu'}\right)\exp\left(L(e^{r^{\mu'}})\right)\right] \\
\text{i.e.,}\quad & \log^{[2]} T(r, fog) \geq \log\left[(\lambda_f^{L^*} - \epsilon)\log\left\{\exp\left(r^{\mu'}\right)\exp\left(L(e^{r^{\mu'}})\right)\right\}\right] \\
\text{i.e.,}\quad & \log^{[2]} T(r, fog) \geq \log(\lambda_f^{L^*} - \epsilon) + \log^{[2]}\left[\exp\left(r^{\mu'}\right)\exp\left(L(e^{r^{\mu'}})\right)\right] \\
\text{i.e.,}\quad & \log^{[2]} T(r, fog) \geq O(1) + \log\left[r^{\mu'} + L\left(\exp\left(r^{\mu'}\right)\right)\right].
\end{aligned}$$

So for a sequence of values of r tending to infinity,

$$\begin{aligned}
& \log^{[2]} T\left\{\exp\left(r^{\rho_g^{L^*}}\right), fog\right\} \\
\geq\ & O(1) + \log\left[\exp\left(r^{\rho_g^{L^*}}.\mu'\right) + L\left\{\exp^{[2]}\left(r^{\rho_g^{L^*}}.\mu'\right)\right\}\right]. \qquad (19)
\end{aligned}$$

Again in view of Lemma 5, we obtain for all sufficiently large values of r that

$$\begin{aligned}
\log T(\exp(r^{\mu}), W(f)) &\leq (\rho_{W(f)}^{L^*} + \epsilon)\log\{\exp(r^{\mu})\exp(L(\exp(r^{\mu})))\} \\
i.e.,\ \log T(\exp(r^{\mu}), W(f)) &\leq (\rho_f^{L^*} + \epsilon)(r^{\mu} + L(\exp(r^{\mu}))). \qquad (20)
\end{aligned}$$

Combining (19) and (20) it follows for a sequence of values of r tending to infinity that

$$\begin{aligned}
& \frac{\log^{[2]} T\left\{\exp\left(r^{\rho_g^{L^*}}\right), fog\right\}}{\log T(\exp(r^{\mu}), W(f))} \\
\geq\ & \frac{O(1) + \log\left[\exp\left(r^{\rho_g^{L^*}}.\mu'\right) + L\left\{\exp^{[2]}\left(r^{\rho_g^{L^*}}.\mu'\right)\right\}\right]}{(\rho_f^{L^*} + \epsilon)\left[r^{\mu} + L(\exp(r^{\mu}))\right]} \\
=\ & \frac{O(1) + \log\left[\exp\left(r^{\rho_g^{L^*}}.\mu'\right)\left\{1 + \frac{\log L\left(\exp^{[2]}\left(r^{\rho_g^{L^*}}.\mu'\right)\right)}{\exp\left(r^{\rho_g^{L^*}}.\mu'\right)}\right\}\right]}{(\rho_f^{L^*} + \epsilon)(r^{\mu}) + (\rho_f^{L^*} + \epsilon)L(\exp(r^{\mu}))} \\
=\ & \frac{O(1) + \log\left\{\exp\left(r^{\rho_g^{L^*}}.\mu'\right)\right\} + \log\left[1 + \frac{\log L\left\{\exp^{[2]}\left(r^{\rho_g^{L^*}}.\mu'\right)\right\}}{\exp\left(r^{\rho_g^{L^*}}.\mu'\right)}\right]}{(\rho_f^{L^*} + \epsilon)r^{\mu} + (\rho_f^{L^*} + \epsilon)L(\exp(r^{\mu}))} \\
=\ & \frac{O(1) + r^{\rho_g^{L^*}}.\mu' + \log\left[1 + \frac{\log L\left\{\exp^{[2]}\left(r^{\rho_g^{L^*}}.\mu'\right)\right\}}{\exp\left(r^{\rho_g^{L^*}}.\mu'\right)}\right]}{(\rho_f^{L^*} + \epsilon)r^{\mu} + (\rho_f^{L^*} + \epsilon)\log L(\exp(r^{\mu}))}. \qquad (21)
\end{aligned}$$

Since $\mu < \rho_g^{L^*}$ we get from (21) that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T\left(\exp\left(r^{\rho_g^{L^*}}\right), fog\right)}{\log T\left(\exp\left(r^{\mu}\right), W(f)\right)} = \infty.$$

This proves the theorem. ■

Theorem 7 *Let f be rational and g be transcendenal meromorphic satisfying $0 < \overline{\lambda}_{fog}^{L^*} \le \overline{\rho}_{fog}^{L^*} < \infty$, $0 < \overline{\lambda}_g^{L^*} < \overline{\rho}_g^{L^*} < \infty$ and $\sum_{a\neq\infty} \delta(a,g) + \delta(\infty, g) = 2$. Then for any positive number A*

$$\begin{aligned} \frac{\overline{\lambda}_{fog}^{L^*}}{A\overline{\rho}_g^{L^*}} &\le \liminf_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, W(g))} \\ &\le \frac{\overline{\lambda}_{fog}^{L^*}}{A\overline{\lambda}_g^{L^*}} \le \limsup_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, W(g))} \le \frac{\overline{\rho}_{fog}^{L^*}}{A\overline{\lambda}_g^{L^*}}. \end{aligned}$$

Proof. From the definition of hyper L*-order and hyper L*-lower order and by Lemma 6 we get for arbitrary positive ϵ and for all sufficiently large values of r,

$$\log^{[2]} T(r, fog) \ge \left(\overline{\lambda}_{fog}^{L^*} - \epsilon\right) \log\left[re^{L(r)}\right] \tag{22}$$

$$\text{and} \quad \log^{[2]} T(r^A, W(g)) \le \left(\overline{\rho}_{W(g)}^{L^*} + \epsilon\right) \log\left[r^A e^{L(r^A)}\right]$$

$$i.e., \quad \log^{[2]} T(r^A, W(g)) \le \left(\overline{\rho}_g^{L^*} + \epsilon\right) \left(A \log r + L(r^A)\right). \tag{23}$$

Combining (22) and(23), we obtain for all sufficiently large values of r that

$$\begin{aligned} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, W(g))} &\ge \frac{\left(\overline{\lambda}_{fog}^{L^*} - \epsilon\right) \log(re^{L(r)})}{\left(\overline{\rho}_g^{L^*} + \epsilon\right)\left(A \log r + L(r^A)\right)} \\ &= \frac{\left(\overline{\lambda}_{fog}^{L^*} - \epsilon\right) \left(\log r + L(r)\right)}{A\left(\overline{\rho}_g^{L^*} + \epsilon\right) \log r + \left(\overline{\rho}_g^{L^*} + \epsilon\right) L(r^A)} \\ &= \frac{\left(\overline{\lambda}_{fog}^{L^*} - \epsilon\right) \log r + \left(\overline{\lambda}_{fog}^{L^*} - \epsilon\right) L(r)}{A\left(\overline{\rho}_g^{L^*} + \epsilon\right) \log r + \left(\overline{\rho}_g^{L^*} + \epsilon\right) L(r^A)}. \end{aligned}$$

Since $\epsilon\,(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, W(g))} \ge \frac{\overline{\lambda}_{fog}^{L^*}}{A\overline{\rho}_g^{L^*}}. \tag{24}$$

Again for a sequence of values of r tending to infinity,

$$\log^{[2]} T(r, fog) \le \left(\overline{\lambda}_{fog}^{L^*} + \epsilon\right) \log\left[re^{L(r)}\right]. \tag{25}$$

Also in view of Lemma 6, we have for all sufficiently large values of r that

$$\log^{[2]} T(r^A, W(g)) \geq \left(\overline{\lambda}_{W(g)}^{L^*} - \epsilon\right) \log\left[r^A e^{L(r^A)}\right]$$

$$i.e., \quad \log^{[2]} T(r^A, W(g)) \geq \left(\overline{\lambda}_{g}^{L^*} - \epsilon\right) \left(A \log r + L(r^A)\right). \tag{26}$$

Combining (25) and (26) we get for a sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, W(g))} &\leq \frac{\left(\overline{\lambda}_{fog}^{L^*} + \epsilon\right) \log\left[re^{L(r)}\right]}{\left(\overline{\lambda}_{g}^{L^*} - \epsilon\right) \left(A \log r + L(r^A)\right)} \\ &= \frac{\left(\overline{\lambda}_{fog}^{L^*} + \epsilon\right) \left(\log r + L(r)\right)}{\left(\overline{\lambda}_{g}^{L^*} - \epsilon\right) \left(A \log r + L(r^A)\right)} \\ &= \frac{\left(\overline{\lambda}_{fog}^{L^*} + \epsilon\right) \log r + \left(\overline{\lambda}_{fog}^{L^*} + \epsilon\right) L(r)}{A\left(\overline{\lambda}_{g}^{L^*} - \epsilon\right) \log r + \left(\overline{\lambda}_{g}^{L^*} - \epsilon\right) L(r^A)}. \end{aligned}$$

As $\epsilon\,(> 0)$ is arbitrary it follows from above that

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, W(g))} \leq \frac{\overline{\lambda}_{fog}^{L^*}}{A\overline{\lambda}_{g}^{L^*}}. \tag{27}$$

Also for a sequence of values of r tending to infinity and by Lemma 6,

$$\log^{[2]} T(r^A, W(g)) \leq \left(\overline{\lambda}_{W(g)}^{L^*} + \epsilon\right) \log\left[r^A e^{L(r^A)}\right]$$

$$\text{i.e.,} \quad \log^{[2]} T(r^A, W(g)) \leq \left(\overline{\lambda}_{g}^{L^*} + \epsilon\right) \log\left[r^A e^{L(r^A)}\right]. \tag{28}$$

Combining (22) and (28) we have for a sequence of values of r tending to infinity

$$\begin{aligned} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, W(g))} &\geq \frac{\left(\overline{\lambda}_{fog}^{L^*} - \epsilon\right) \log\left[re^{L(r)}\right]}{\left(\overline{\lambda}_{g}^{L^*} + \epsilon\right) \left(\log r^A e^{L(r^A)}\right)} \\ &= \frac{\left(\overline{\lambda}_{fog}^{L^*} - \epsilon\right) \left(\log r + L(r)\right)}{\left(\overline{\lambda}_{g}^{L^*} + \epsilon\right) \left(A \log r + L(r^A)\right)} \\ &= \frac{\left(\overline{\lambda}_{fog}^{L^*} - \epsilon\right) \log r + \left(\overline{\lambda}_{fog}^{L^*} - \epsilon\right) L(r)}{A\left(\overline{\lambda}_{g}^{L^*} + \epsilon\right) \log r + \left(\overline{\lambda}_{g}^{L^*} + \epsilon\right) L(r^A)}. \end{aligned}$$

Since $\epsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, W(g))} \geq \frac{\overline{\lambda}_{fog}^{L^*}}{A\overline{\lambda}_{g}^{L^*}}. \tag{29}$$

Also for all sufficiently large values of r,

$$\log^{[2]} T(r, fog) \leq \left(\overline{\rho}_{fog}^{L^*} + \epsilon\right) \log\left[re^{L(r)}\right]. \tag{30}$$

From (26) and (30) we obtain for all sufficiently large values of r,

$$\begin{aligned}
\frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, W(g))} &\leq \frac{\left(\overline{\rho}_{fog}^{L^*} + \epsilon\right) \log(re^{L(r)})}{\left(\overline{\lambda}_g^{L^*} - \epsilon\right) \log(r^A e^{L(r^A)})} \\
\text{i.e.,}\quad \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, W(g))} &\leq \frac{\left(\overline{\rho}_{fog}^{L^*} + \epsilon\right) (\log r + L(r))}{\left(\overline{\lambda}_g^{L^*} - \epsilon\right) (\log r^A + L(r^A))} \\
\text{i.e.,}\quad \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, W(g))} &\leq \frac{\left(\overline{\rho}_{fog}^{L^*} + \epsilon\right) \log r + \left(\overline{\rho}_{fog}^{L^*} + \epsilon\right) L(r)}{A\left(\overline{\lambda}_g^{L^*} - \epsilon\right) \log r + \left(\overline{\lambda}_g^{L^*} - \epsilon\right) L(r^A)}.
\end{aligned}$$

Since $\epsilon\,(>0)$ is arbitrary it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, W(g))} \leq \frac{\overline{\rho}_{fog}^{L^*}}{A\overline{\lambda}_g^{L^*}}. \tag{31}$$

Thus the theorem follows from (24),(27),(29) and (31). ■

Theorem 8 *Let f be meromorphic and g be transcendental entire such that $(i) 0 < \rho_g^{L^*} < \infty, (ii)\ \sigma_g^{L^*} > 0, (iii) 0 < \rho_{fog}^{L^*} < \infty, (iv) \sigma_{fog}^{L^*} < \infty, (v) \rho_f^* < 1$ and $(vi) \sum_{a\neq\infty} \delta(a, g) + \delta(\infty, g) = 2$. Then*

$$\liminf_{r\to\infty} \frac{\log T(r, fog)}{\log T(r, W(g))} = 0.$$

Proof. From the definition of L*-type we have for arbitrary positive ϵ and for all sufficiently large values of r,

$$\log T(r, fog) \leq \left(\sigma_{fog}^{L^*} + \epsilon\right) \left(re^{L(r)}\right)^{\rho_{fog}^{L^*}}. \tag{32}$$

Again in view of Lemma 5, we get for a sequence of values of r tending to infinity that

$$\begin{aligned}
T(r, W(g)) &\geq \left(\sigma_{W(g)}^{L^*} - \epsilon\right) \left(re^{L(r)}\right)^{\rho_{W(g)}^{L^*}} \\
\text{i.e.,}\quad T(r, W(g)) &\geq \left[\{1 + k - k\delta(\infty, g)\}\, \sigma_g^{L^*} - \epsilon\right] \left(re^{L(r)}\right)^{\rho_g^{L^*}}.
\end{aligned} \tag{33}$$

Since $\rho_{fog}^{L^*} < \infty$, it follows that $\rho_f^{L^*} = 0\,\{cf.[2]\}$. So in view of Lemma 7, from (32) and (33) we obtain for a sequence of values of r tending to infinity,

$$\frac{T(r, fog)}{T(r, W(g))} \leq \frac{\left(\sigma_{fog}^{L^*} + \epsilon\right) \left(re^{L(r)}\right)^{\rho_{fog}^{L^*}}}{\left[\{1 + k - k\delta(\infty, g)\}\, \sigma_g^{L^*} - \epsilon\right] \left(re^{L(r)}\right)^{\rho_g^{L^*}}}$$

$$\text{i.e.,}\quad \frac{T(r,fog)}{T(r,W(g))} \le \frac{\left(\sigma_{fog}^{L^*}+\epsilon\right)\left(re^{L(r)}\right)^{\rho_f^*.\rho_g^{L^*}}}{\left[\{1+k-k\delta(\infty,g)\}\,\sigma_g^{L^*}-\epsilon\right]\left(re^{L(r)}\right)^{\rho_g^{L^*}}}$$
$$= \frac{\left(\sigma_{fog}^{L^*}+\epsilon\right)\left(re^{L(r)}\right)^{(\rho_f^*-1).\rho_g^{L^*}}}{\left[\{1+k-k\delta(\infty,g)\}\,\sigma_g^{L^*}-\epsilon\right]}$$

$$\text{i.e.,}\quad \frac{T(r,fog)}{T(r,W(g))} \le \frac{\left(\sigma_{fog}^{L^*}+\epsilon\right)r^{(\rho_f^*-1).\rho_g^{L^*}}\left(e^{L(r)}\right)^{(\rho_f^*-1).\rho_g^{L^*}}}{\left[\{1+k-k\delta(\infty,g)\}\,\sigma_g^{L^*}-\epsilon\right]}.$$

Since $\epsilon\,(>0)$ is arbitrary in view of condition (v) it follows that

$$\liminf_{r\to\infty}\frac{\log T(r,fog)}{\log T(r,W(g))}=0.$$

This proves the theorem. ■

References

[1] Bergweiller, W.: On the Nevanlinna Characteristic of a composite function, Complex Variables, Vol. 10 (1988), pp. 225-236.

[2] Bergweiller, W.: Order and lower order of composite meromorphic functions, Michigan Math. J. Vol. 36 (1989), pp. 135-146.

[3] Bergweiler, W.: On the growth rate of composite meromorphic functions, Complex Variables, Vol. 14 (1990), pp. 187-196.

[4] Clunie, J.: The composition of entire and meromorphic functions, Mathematical essays dedicated to A.J.Macintyre, Ohio University Press, (1970), pp. 75-92.

[5] Lahiri, I. and Banerjee, A.: Value distribution of a wronskian, Portugalie Mathematica, Vol. 61, Fasc.2 (2004), Nova Série, pp. 161-175.

[6] Liao, L. and Yang, C.C.: On the growth of composite entire functions, Yokohama Math. J., Vol. 46(1999), pp. 97-107.

[7] Somasundaram, D. and Thamizharasi, R.: A note on the entire functions of L-bounded index and L-type, Indian J. Pure and Appl. Math., Vol. 19, No. 3 (March 1988), pp. 284-293.