FIXED AND COINCIDENCE POINT RESULTS ON METRIC SPACE

T. Som and *Amalendu Choudhury
Department of Applied Mathematics, Institute of Technology
Banaras Hindu University, Varanasi -221005, India
*Department of Mathematics and Statistics
Haflong Govt. College, Haflong, Dima Hasao-788819
Assam, India

Abstract : *The first result of the present paper deals with few common fixed point for weakly compatible mappings of type (A) and satisfying some general inequality condition involving product terms. The results obtained generalize the earlier results of Fisher et. al.(1987), Nesic (1992), Tas et. al.(1996) and others in turn. The second result on coincidence point generalizes the result of Imdad and Kumar (2005).*

Key words: Weakly compatible map, coincidence point, common fixed point.

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1. Introduction

The idea of common fixed point for commuting mappings was initially given by Jungck [4], which gave an initial thrust for further generalizations of the theory of common fixed point. Many results were obtained by several mathematicians viz., Hadzic [2], Pathak [10], Yeh [15] etc. The commutativity condition of mappings was further replaced by a weaker type of notion viz., weakly commuting mappings introduced by Sessa [11]. Several common fixed point theorems have been proved for such mappings by many authors viz., Sessa et. al. [12], Fisher and Sessa [1] and others. The notion of weak commutativity has been further weakened by the notion of compatible mappings, introduced by Jungck [5], compatible mappings of type (A) by Jungck et al [6], which gave a new direction towards more comprehensive results in the context of common fixed point theory. The beauty of these properties are that compatibility implies weakly commuting and commuting imlpies weakly commuting but the converse of such notions need not be true. Here we give some preliminary definitions and results.

Definition 1.1 (Sessa [11]) Two self mappings A and S of a metric space (X, d) are called weakly commuting if $d(ASx, SAx) \leq d(Ax, Sx)$ for every $x \in X$.

Definition 1.2 (Jungck [5]) Let f and g be two self maps of a metric space (X, d), then f and g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.

Definition 1.3 Two self mappings S and T of a metric space (X, d) are said to be compatible of type (A) if $\lim_{n\to\infty} d(TS(x_n),SS(x_n)) = 0$ and $\lim_{n\to\infty} d(ST(x_n),TT(x_n)) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t$ for some $t \in X$.

Definition 1.4 The self mappings S and T of a metric space (X, d) are said to be weakly compatible of type (A) if $\lim_{n\to\infty} d(STx_n, TTx_n) \le d(Sz,Tz) \le \lim_{n\to\infty} d(Tz, TTx_n)$

and $\lim_{n\to\infty} d(TSx_n ,SSx_n) \le d(Sz,Tz) \le \lim_{n\to\infty} d(Sz, SSx_n)$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Example 1.1 Let $X = [0,1]$ be a metric space with the usual metric $d(x, y) = |x - y|$. Define the mappings $S ,T : X \to X$ by

$$Sx = \begin{cases} x & \text{for } x \in [0,\frac{1}{2}] \\ 1 & \text{for } x \in [\frac{1}{2},1] \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1-x & \text{for } x \in [0,\frac{1}{2}] \\ 1 & \text{for } x \in [\frac{1}{2},1] \end{cases}$$

respectively. Then S and T are not continuous at $z = ½$. Now we assert that S and T are not compatible but they are both compatible of type (A) and weakly compatible of type (A) . Suppose that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = z$. By the definition of S and T, $z \in \{½ ,1\}$. Since S and T agree on [½ ,1], we need to consider $z = ½$ and so we can suppose that $x_n \to ½$ as $n \to \infty$ and $x_n < ½$ for $n = 1,2,3,\ldots$..

Then we have, $Sx_n = x_n \to ½$ (from the left) and $Tx_n = (1- x_n) \to ½$ (from the right)

$ST x_n = S (1 - x_n) = 1$; $TSx_n = Tx_n = (1 - x_n) \to ½$.

Thus it follows that $\lim_{n\to\infty} d(STx_n ,TSx_n) = \lim_{n\to\infty} | 1 - ½ | = ½ \neq 0$

But $\lim_{n\to\infty} d(STx_n ,TTx_n) = \lim_{n\to\infty} | STx_n - TTx_n| = \lim_{n\to\infty} | 1 - T(1- x_n) | = 1 - 1 = 0.$

$\lim_{n\to\infty} d(TSx_n ,SSx_n) = \lim_{n\to\infty} | T Sx_n - SSx_n| = |(1 -x_n) - x_n | = 1 - 1 = 0.$

$d(Sz, Tz) = | S(1/2) - T(1/2) | = 1 - 1 = 0.$

$\lim_{n\to\infty} d(Tz,TTx_n) = \lim_{n\to\infty} |T(1/2) - T(1 - x_n)| = 1 - 1 = 0$;

$\lim_{n\to\infty} d(Sz, SSx_n) = \lim_{n\to\infty} | S(1/2) - x_n| = |1 - ½ | = ½$.

Therefore, S and T are both compatible of type (A) and weakly compatible of type (A) but they are not compatible.

Example 1.2 Let $X = [0,\infty)$ be a metric space with the usual metric $d(x,y) = | x - y |$. Define the mappings $S,T : X \to X$ by

$$Sx = \begin{cases} \frac{1}{2} + x & \text{for } x \in \left[0,\frac{1}{2}\right] \\ 1 & \text{for } x = \frac{1}{2} \\ 0 & \text{for } x \in \left(\frac{1}{2},\infty\right) \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{1}{2} - x & \text{for } x \in \left[0,\frac{1}{2}\right] \\ 2 & \text{for } x = \frac{1}{2} \\ 0 & \text{for } x \in \left(\frac{1}{2},\infty\right) \end{cases}$$

respectively. Then clearly S and T are not continuous at $z = ½$ and the maps S and T are both compatible and weakly compatible of type (A) but they are not compatible of type (A).

Proposition 1.1 (Jungck et.al [6]) Let S and T be continuous mappings from a metric space (X,d) into itself. Then S and T are compatible if and only if they are compatible of type (A).

Let (X,d) be a metric space and let T be a mapping from X into itself. We say that a metric space (X,d) is T- orbitally complete if every Cauchy sequence of the form $\{T^i x\}_{i\in N}$ for $x \in X$ converges to a point in X. For such orbitally complete metric space, Nesic [9] proved the following result:

Theorem 1.1 Let (X,d) be a metric space and let T be a self mapping of X satisfying the following inequality condition :

$$[\, 1+ pd(x,y) \,]\, d(Tx,Ty) \leq p\, [\, d(x,Tx)\, d(y,Ty) + d(x,Ty)\, d(y,Tx)] + q \max \{d(x,y), d(x,Tx), d(y,Ty), \tfrac{1}{2}\, [\, d(x,Ty) + d(y,Tx)]\} \qquad \text{(i)}$$

for all $x, y \in X$, where $p \geq 0$ and $0 < q < 1$. If (X,d) is T- orbitally complete, then T has a unique fixed point in X.

Proposition 1.2 (Lal et al [5]) Let S and T be mappings of a metric space (X,d) into itself such that $d(Sz,Tz) \leq \lim_{n\to\infty} d(Tz,TTx_n)$ and $d(Sz,Tz) \leq \lim_{n\to\infty} d(Sz,SSx_n)$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n Sx_n = \lim_n Tx_n = z$ for some $z \in X$. If S and T are compatible of type (A), then they are weakly compatible of type (A), but the converse is not true.

Proposition 1.3 (Murthy et al [8]) Let S and T be continuous mappings from a metric space (X,d) into itself. If S and T are weakly compatible of type (A), then they are compatible of type (A).

Proposition 1.4 (Lal et al [7]) Let S and T be weakly compatible mappings of type (A) from the metric space (X, d) into itself. If $Sz = Tz$ for some $z \in X$ then

$$STz = TTz = TSz = SSz.$$

Proposition 1.5 (Murthy et.al [8]) Let S and T be weakly compatible mappings of type (A) from the metric space (X,d) into itself and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$, then we have the following

(i) $\lim_{n\to\infty} TSx_n = Sz$ if S is continuous.

(ii) $\lim_{n\to\infty} STx_n = Tz$ if T is continuous.

(iii) $STz = TSz$ if S and T are continuous.

Motivated by the result of Nesic [9] and Fisher et. al.[1] , we establish some common fixed point results for weakly compatible mappings of type (A).

Lemma 1.1 (Singh et al [13]) For all $t > 0$, $\Gamma(t) < t$ if and only if

$$\lim_{n\to\infty} \Gamma^n(t) = 0$$

where Γ^n denotes the n-times composition of Γ.

2. Main Results

In what follows we give our result, which extend and improve the results of Fisher et al [1], Nesic [9], Tas et al [14] and many others in turn.

Theorem 2.1 Let A, B, S and T be four self mappings of a complete metric space (X, d) satisfying the following conditions :

(ii) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(iii) the pairs {A,S} and {B,T} are weakly compatible of type (A),

(iv) one of A, B, S and T is continuous,

(v) $[1 + pd(Sx,Ty)]\, d(Ax,By) \leq p \max \{ d(Sx,Ax)\, d(Ty,By), d(Sx,By)\, d(Ty,Ax),$

$d(Sx,Ty)\, d(Ax,By)\} + q \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), d(Sx,By), d(Ax,Ty)\}$

for all $x,y \in X$, where $q < 1$ and $p \geq 0$.

Then A, B, S and T have a unique common fixed point in X.

Proof : Since $A(X) \subset T(X)$, for an arbitrary point $x_o \in X$, we can choose a point $x_1 \in X$

such that $y_o = Ax_o = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $y_1 = Bx_1 = Sx_2$ and so on . Inductively, we get a sequence $\{y_n\}$ in X such that

(vi) $$y_{2n} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$$

for $n = 0,1,2,..$

For simplicity, let $\alpha_{2n} = d(Ax_{2n},Bx_{2n+1})$

and $\alpha_{2n+1} = d(Bx_{2n+1}, Ax_{2n+2})$, for $n = 0,1,2,\ldots$.

Now from (v), we have

$$\begin{aligned}[1 + pd(Sx_{2n},Tx_{2n+1})]\, d(Ax_{2n},Bx_{2n+1} &\leq p \max \{d(Sx_{2n},Ax_{2n})d(Tx_{2n+1},Bx_{2n+1}),\\ &\quad d(Sx_{2n},Bx_{2n+1})\, d(Tx_{2n+1},Ax_{2n}),\, d(Sx_{2n},Tx_{2n+1})\, d(Ax_{2n},Bx_{2n+1})\}\\ &\quad + q \max \{d(Sx_{2n},Tx_{2n+1}),\, d(Sx_{2n},Ax_{2n}),\, d(Tx_{2n+1},Bx_{2n+1}),\\ &\quad d(Sx_{2n},Bx_{2n+1}),\, d(Ax_{2n},Tx_{2n+1})\}\end{aligned}$$

or, $$\begin{aligned}[1+pd(Ax_{2n},Bx_{2n-1})]\, d(Ax_{2n},Bx_{2n+1}) &\leq p \max \{d(Ax_{2n},Bx_{2n-1})\, d(Ax_{2n},Bx_{2n+1}),\\ &\quad d(Bx_{2n-1},Bx_{2n+1})\, d(Ax_{2n},Ax_{2n}),\, d(Bx_{2n-1},Ax_{2n})\, d(Ax_{2n},Bx_{2n+1})\}\\ &\quad + q \max \{d(Ax_{2n},Bx_{2n-1}),\, d(Ax_{2n},Bx_{2n-1}),\, d(Ax_{2n},Bx_{2n+1}),\\ &\quad d(Bx_{2n-1},Bx_{2n+1}),\, d(Ax_{2n},Ax_{2n})\}\end{aligned}$$

or, $[1+ p\,\alpha_{2n-1}]\, \alpha_{2n} \leq p \max \{\alpha_{2n-1}\, \alpha_{2n} , \alpha_{2n-1}\, \alpha_{2n}\} + q \max \{\alpha_{2n-1}, \alpha_{2n-1}, \alpha_{2n}, (\alpha_{2n-1} + \alpha_{2n}), 0\}$

which implies that,

$$[1 + p\, \alpha_{2n-1}]\alpha_{2n} \leq p\ \alpha_{2n-1}\, \alpha_{2n} + q \max\{ \alpha_{2n-1}, \alpha_{2n-1}, \alpha_{2n} ,(\alpha_{2n-1} + \alpha_{2n}) , 0\}$$

If $\alpha_{2n} > \alpha_{2n-1}$ for some n, then we have $\alpha_{2n} \leq q\alpha_{2n} < \alpha_{2n}$, which is a contradiction. Thus we have, $\alpha_{2n} \leq q\alpha_{2n-1}$ for $n = 1,2,\ldots$. Similarly we have $\alpha_{2n+1} \leq q\alpha_{2n}$.

Proceeding in this manner, we get

$$\alpha_{2n} \leq q\alpha_{2n-1} \leq \ldots\ldots\ldots\ldots\leq q^{2n}\, \alpha_o \to 0 \quad \text{as } n \to \infty.$$

Thus $\{y_{2n}\}$ is a Cauchy sequence in X and so $\{Ax_{2n}\}$ is a Cauchy sequence in X. Similarly we can show that $\{Bx_{2n+1}\}$ is also a Cauchy sequence in X. Since (X, d) is complete, the sequence $\{y_{2n}\}$ defined by (vi) converges to a limit $z \in X$.

Thus, the subsequences $\{Ax_{2n}\} = \{Tx_{2n+1}\}$,$\{Bx_{2n+1}\} = \{Sx_{2n+2}\}$ of $\{y_{2n}\}$ also converge to z. Now suppose that S is continuous .Then

$$SAx_{2n} \to Sz \quad \text{as} \quad n \to \infty.$$

Since A and S are weakly compatible mappings of type (A), by proposition 5

$$ASx_{2n} \to Sz \quad \text{as} \quad n \to \infty.$$

Replacing x by Sx_{2n} and y by x_{2n+1} in (v), we have

$$[1+ pd(S^2x_{2n}, Tx_{2n+1})]\, d(ASx_{2n},Bx_{2n+1}) \leq p \max \{d(S^2x_{2n},ASx_{2n})\, d(Tx_{2n+1},Tx_{2n+1}),$$
$$d(S^2x_{2n},Bx_{2n+1})\, d(Tx_{2n+1},ASx_{2n}),\ d(S^2x_{2n},Tx_{2n+1})\, d(ASx_{2n},Bx_{2n+1})\}$$
$$+ q \max\{d(S^2x_{2n},Tx_{2n+1}),\ d(S^2x_{2n},ASx_{2n}),\ d(Tx_{2n+1},Bx_{2n+1}),$$
$$d(S^2x_{2n},Bx_{2n+1}),\ d(ASx_{2n}, Tx_{2n+1})\}$$

Taking limit as $n \to \infty$, we have $d(Sz, z) \leq qd(Sz, z)$.

Hence $(1 - q)\, d(Sz, z) \leq 0$. Therefore, $Sz = z$ (since $q < 1$)

Again, replacing x by z and y by x_{2n+1} in (v) and taking limit as $n \to \infty$, we have

$$d(Az, z) \leq q\, d(Az,z),$$

which means that $Az = z$.

Since $A(X) \subset T(X)$, there exists a point u in X such that $Az = Tu = z$.

Again from (v), it follows that :

$$[\, 1 + pd(Sz,Tu)]\, d(Az,Bu) \leq p \max \{ d(Sz, Az)\, d(Tu,Bu),\ d(Sz,Bu)\, d(Tu,Az),\ d(Sz,Tu)$$
$$d(Az,Bu)\} + q \max \{d(Sz,Tu),\ d(Sz,Az),\ d(Tu,Bu),\ d(Sz,Bu),\ d(Az,Tu)\}$$

which implies that,

$$d(z,Bu) \leq qd(z,Bu)$$

and so we have $z = Bu = Tu$. But since B and T are weakly compatible of type (A), by proposition 4 ,

$$Bz = BTu = TTu = Tz.$$

Now by (v), we have

$$[1+ pd(Sz,Tz)]\, d(z,Bz) \leq p \max \{d(Sz,Az)\, d(Tz,Bz),\ d(Sz,Bz)\, d(Tz,Az),$$
$$d(Sz, Tz)d(Az,Bz)\} + q \max\{d(Sz,Tz),\ d(Sz,Az),\ d(Tz,Bz),\ d(Sz,Bz),\ d(Az,Tz)\}$$

which implies that

$$d(z,Bz) \leq qd(z,Bz).$$

Hence $z = Bz = Tz$ and therefore z is a common fixed point of A, B, S and T.

For uniqueness of the common fixed point z, we suppose that z and w $(z \neq w)$ are common fixed points of A, B, S and T.
Using (v), again we have,

$$d(z, w) \leq q\, d(z, w)$$

Hence $(1 - q)\, d(z, w) \leq 0$

implies that $z = w$ for $q < 1$, a contradiction.

Therefore z is the unique common fixed point of A, B, S and T. Similarly we can complete the proof by assuming that T or A or B is continuous in lieu of S being continuous.

As an immediate consequence of Theorem 2.1 with A = B, we have the following corollary.

Corollary 2.1 Let A, S and T be mappings of a complete metric space (X, d) into itself satisfying the following conditions :

(vii) $A(X) \subset S(X) \cap T(X)$

(viii) the pairs $\{A,S\}$ and $\{A,T\}$ are weakly compatible of type (A),

(ix) $$[1 + pd(Sx,Ty)]\, d(Ax,Ay) \leq [\, p \max \{d(Sx,Ax)d(Ty,Ax), d(Sx,Ay)\, d(Ty,Ax)\} + q \max \{d(Sx,Ty), d(Sx,Ax), d(Ty,Ay), d(Sx,Ay), d(Ty,Ax)\}]$$

for all $x,y \in X$ and $p \geq 0$. Then A, S and T have a unique common fixed point in X.

The next theorem generalizes the result of Imdad and Kumar [3] under a modified inequality condition.

Theorem 2.2 Let A, B, S, T, I, F, G and J be self mappings of a metric space (X,d) with $AB(X) \subset GJ(X)$ and $ST(X) \subset FI(X)$ satisfying

(a)
(xiii) $$[1+ p\, d(FIx,GJy)]\, d(ABx,STy)] \leq p \max \{d(FIx,ABx)d(GJy,STy), d(FIx,STy)d(GJy,ABx), d(FIx,GJy)d(ABx,STy)\} + q \max\{d(FIx,GJy), d(FIx,ABx), d(GJy,STy), d(FIx,STy), d(GJy,ABx), d(ABx,STy)\}$$

for all $x,y \in X$, $p \geq 0$, $0 < q < 1$. If one of AB(X), ST(X), GJ(X), FI(X) is complete subspace of X, then

(b) (AB, FI) has a coincidence point.

(c) (ST, GJ) has a coincidence point.

Further, if the pairs (AB, FI) and (ST,GJ) are coincidentally commuting, then AB,ST,FI and GJ have a unique common fixed point z.

Moreover, if the pairs(A,B), (AB,I), (AB,F), (FI,A), (FI,B), (F,I), (S,T), (ST,G), (ST,J), (G,J), (GJ,S), (GJ,T) commute at z, then z becomes a unique common fixed point of A, B, S, T, I, F, G and J.

Proof : Let x_0 be an arbitrary point in X. Since $AB(X) \subset GJ(X)$, we find a point x_1 in X with $ABx_0 = GJx_1$ and since $ST(X) \subset FI(X)$, we also choose a point x_2 in X with $STx_1 = FIx_2$. Using the argument repeatedly, we can construct a sequence $\{x_n\}$ such that

$$z_{2n} = ABx_{2n} = GJx_{2n+1} \quad \text{and}$$

(xiv) $$z_{2n+1} = STx_{2n+1} = FIx_{2n+2}$$

for n = 0,1,2,… Let us put

$$\alpha_{2n} = d(ABx_{2n}, STx_{2n+1}) = d(z_{2n}, z_{2n+1})$$

$$\alpha_{2n+1} = d(STx_{2n+1}, ABx_{2n+2}) = d(z_{2n+1}, z_{2n+2})$$

for n = 0,1,2,…. Now using (xiii),we have

$$[1+ pd(FIx_{2n},GJx_{2n+1})]\, d(ABx_{2n},STx_{2n+1}) \leq p \max \{d(FIx_{2n},ABx_{2n})d(GJx_{2n+1},STx_{2n+1}),$$
$$d(FIx_{2n},STx_{2n+1})d(GJx_{2n+1},ABx_{2n}),\ d(FIx_{2n},GJx_{2n+1})d(ABx_{2n},STx_{2n+1})\}$$
$$+ q \max \{d(FIx_{2n},GJx_{2n+1}),d(FIx_{2n},ABx_{2n}),d(GJx_{2n+1},STx_{2n+1}),d(FIx_{2n},STx_{2n+1}),$$
$$d(GJx_{2n+1},ABx_{2n}),d(ABx_{2n},STx_{2n+1})\}$$

or, $$[1+ pd(z_{2n-1},z_{2n})]d(z_{2n},z_{2n+1}) \leq p \max \{d(z_{2n-1},z_{2n})d(z_{2n},z_{2n+1}),\ d(z_{2n-1},z_{2n+1}),d(z_{2n},z_{2n}),$$
$$d(z_{2n-1},z_{2n})\, d(z_{2n},z_{2n+1}) + q \max \{d(z_{2n-1},z_{2n}),\ d(z_{2n-1},z_{2n}),d(z_{2n},z_{2n+1})d(z_{2n-1},z_{2n+1}),$$
$$d(z_{2n},z_{2n}),d(z_{2n},z_{2n+1})\}$$

or, $$(1 +p\alpha_{2n-1})\, \alpha_{2n} \leq p \max\{\alpha_{2n-1}\alpha_{2n},\ 0,\ \alpha_{2n-1}\, \alpha_{2n}\} + q \max \{\alpha_{2n-1},\ \alpha_{2n-1},\ \alpha_{2n}, \alpha_{2n-1}+ \alpha_{2n},\ 0,\ \alpha_{2n}\}$$

or, $$(1 + p\, \alpha_{2n-1})\, \alpha_{2n} \leq p\, \alpha_{2n-1}\, \alpha_{2n} + q\, (\alpha_{2n-1} + \alpha_{2n})$$

or, $$\alpha_{2n} \leq \frac{q}{1-q}\, \alpha_{2n-1} = r\, \alpha_{2n-1}, \text{ where } r = \frac{q}{1-q}$$

Hence, $$\alpha_{2n} \leq r\, \alpha_{2n-1} \leq \ldots\ldots\ldots\ldots \leq r^{2n-1}\, \alpha_0$$

Therefore the sequence $\{\alpha_{2n}\}$ is Cauchy sequence. That is, $\{ABx_{2n}\}$ and $\{STx_{2n+1}\}$ are Cauchy sequences in X. Now suppose that GJ(X) is a complete subspace of X, then we have a subsequence $\{z_{2n}\}$ of $\{\alpha_{2n}\}$, contained in GJ(X) with a limit z in GJ(X).

Let $u \in (FI)^{-1}(z)$ then $FIu = z$. we also need to use the fact that the subsequence $\{z_{2n-1}\}$ also converges to u, otherwise, let on contrary that $\{z_{2n-1}\}$ converges to z' then using (xiii) we get,

$$[1+pd(z_{2n-1},z_{2n})]\, d(z_{2n},z_{2n+1}) \le p \max \{d(z_{2n-1},z_{2n})d(z_{2n},z_{2n+1}), d(z_{2n-1},z_{2n+1})d(z_{2n},z_{2n}), d(z_{2n-1},z_{2n})\ d(z_{2n},z_{2n+1}) + q \max \{d(z_{2n-1},z_{2n}), d(z_{2n-1},z_{2n}), d(z_{2n},z_{2n+1}), d(z_{2n-1},z_{2n+1}), d(z_{2n},z_{2n}), d(z_{2n},z_{2n+1})\}$$

which, on letting $n \to \infty$ reduces to

$$[1 + pd(z,z')]\, d(z,z') \le p \max \{d(z,z')d(z,z'), 0, d(z,z')d(z,z')\} + q \max \{ d(z,z'), d(z,z'), d(z,z'), d(z,z'), 0, d(z,z') \}$$

or, $$d(z,z') + pd(z,z')d(z,z') \le p\, d(z,z')\, d(z,z') + q\, d(z,z')$$

or, $$(1 - q)\, d(z, z') \le 0$$ which implies that $z = z'$ as $q < 1$.

Now, we prove that $ABu = z$, putting $x = u$ and $y = x_{2n-1}$ in (xiii), we get

$$[1 + pd(FIu,GJx_{2n-1})]d(ABu,STx_{2n-1}) \le p \max \{d(FIu,ABu)d(GJx_{2n-1},STx_{2n-1}), d(FIu,STx_{2n-1})d(GJx_{2n-1},ABu), d(FIu,GJx_{2n-1})\, d(ABu,STx_{2n-1})\} + q \max\{d(FIu,GJx_{2n-1}), d(FIu,ABu), d(GJx_{2n-1},STx_{2n-1}), d(FIu,STx_{2n-1}), d(GJx_{2n-1},ABu), d(ABu,STx_{2n-1})\},$$

which on letting $n \to \infty$ it becomes

$$[1 + pd(z,z)]\, d(ABu,z) \le p \max \{ d(z,ABu)d(z,z), d(z,z)d(z,ABu), d(z,z)d(ABu,z)\} + q \max\{d(z,z), d(z,ABu), d(z,z), d(z,z), d(z,ABu), d(ABu,z)\}$$

or, $$d((ABu,z) \le p \max \{0, 0, 0\} + q \max \{0, d(ABu,z), 0,0, d(ABu,z), d(ABu,z)\}$$

or, $$(1 - q)\, d(ABu, z) \le 0$$

Therefore, $ABu = z = FIu$, since $q < 1$. This establishes (b). Again since $AB(X) \subset GJ(X)$, $ABu = u$ implies that $z \in GJ(X)$. Let $u \in (GJ)^{-1}z$ then $GJu = z$.

Again using earlier arguments, it can be easily shown that $STu = z$ yielding thereby

$$GJu = STu = z.$$

which establishes (c).

We now assume that FI(X) is a complete subspace of X then analogous arguments establishes (b) and (c) . If ST(X) is complete, then $z \in ST(X) \subset FI(X)$ and in case AB(X) is complete, then

$$z \in AB(X) \subset GJ(X) .$$

Thus (b) and (c) are completely established.

Moreover, if the pairs (AB,FI) and (ST,GJ) are coincidentally commuting at u and v respectively, then

$$z = ABu = FIu = STu = GJu$$

$$ABz = AB(FIu) = FI(ABu) = FIz$$

$$STz = ST(GJu) = GJ(STu) = GJz.$$

If $STz \neq z$, then using (xiii), we get

$$[1+ pd(FIu,GJz)]\, d(ABu,STz) \leq p \max \{d(FIu,ABu)d(GJz,STz), d(FIu,STz)d(GJz,ABu), d(FIu,GJz)\, d(ABu,STz)\} + q \max\{d(FIu,GJz), d(FIu,ABu), d(GJz,STz), d(FIu,STz), d(GJz,ABu), d(ABu,STz)\}$$

or, $$[1 + pd(z,STz)]\, d(z,STz) \leq p \max \{d(z,z)d(GJz,GJz), d(z,STz)d(z,STz), d(z,STz)d(z,STz) + q \max\{d(z,STz), d(z,z), d(GJz,GJz), d(z,STz), d(z,STz), d(z,STz)\}$$

or, $$d(z,STz) + pd(z,STz)\, d(z,STz) \leq p\, d(z,STz)d(z,STz) + q\, d(z,STz)$$

or, $$(1 - q)\, d(z,STz) \leq 0.$$

Therefore, $z = STz$ as $q < 1$.

Similarly we can show that z = ABz. Thus z is a common fixed point of AB, ST, FI, GJ.

The uniqueness of common fixed point follows easily from (xiii). Also z remains the unique common fixed point of both the pairs separately.

Now using the commutativity of various pairs at z, we can write

$$Az = A(ABz) = AB(Az) ,\ Az = A(FIz) = F(Az) \ ;$$

$$Bz =B(ABz) = AB(Bz) ,\ Bz = B(FIz) = FI(Bz)$$

$$Iz = I(ABz) = AB(Iz), \quad Iz = I(FIz) = FI(Iz) \;;$$

$$Fz = F(ABz) = AB(Fz), \quad Fz = F(FIz) = FI(Fz)$$

$$Sz = S(STz) = ST(Sz), \quad Sz = S(GJz) = GJ(Sz) \;;$$

$$Tz = T(STz) = ST(Tz), \quad Tz = T(GJz) = GJ(Tz)$$

$$Gz = G(STz) = ST(Gz), \quad Gz = G(GJz) = GJ(Gz);$$

$$Jz = J(STz) = ST(Jz), \quad Jz = J(GJz) = GJ(Jz)$$

which shows that Az, Bz, Iz, Fz are the common fixed point for the pair (AB, FI) where as Sz, Tz, Gz and Jz are common fixed point of the pair (ST,GJ).
Now uniqueness of common fixed point of both the pairs, we can conclude that

$$z = Az = Bz = Fz = Iz = Sz = Tz = Gz = Jz.$$

which shows that z remains the unique fixed point of A, B, S, T, I, F, G and J. This completes the proof.

By setting F = G = Identity map, we have the following corollary:

Corollary 2.1 Let A, B, S, T, I and J be self mappings of a metric space (X,d) with

$AB(X) \subset J(X)$, $ST(X) \subset I(X)$, satisfying

(A) $$[\,1 + pd(Ix,Jy)]\, d(ABx,STy) \le p \max\{\, d(Ix,ABx)d(Jy,STy),\, d(Ix,STy)d(Jy,ABx)\}$$
$$+\, q \max\{\, d(Ix,Jy),\, d(Ix,ABx),\, d(Jy,STy),\, d(Ix,STy),\, d(Jy,ABx)\}$$

for all $x,y \in X$ with $p \ge 0$. If one of AB(X), ST(X), I(X), J(X) is a complete subspace of X, then

(B) (AB, I) has a coincidence point.

(C) (ST, J) has a coincidence point.

Further, if the pairs (AB, I) and (ST, J) are coincidentally commuting, then AB, ST, I, J have a unique common fixed point .
Moreover if the pairs (A, B), (AB, I),(A, I),(B, I),(S, T),(ST, J),(S, J) and (T,J) commute at z, then z becomes a unique common fixed point of A, B, S, T, I and J .

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