GROWTH RATES OF WRONSKIANS GENERATED BY ITERATED ENTIRE FUNCTIONS ON THE BASIS OF THEIR MAXIMUM TERMS

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Abstract

In this paper we investigate the comparative growth properties of maximum terms of composite entire functions and wronskians generated by one of the factors.

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1 Introduction, Definitions and Notations.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum term $\mu(r,f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu(r,f) = \max_{n\geq 0}(|a_n|\, r^n)$.

To start our paper we just recall the following definitions :

Definition 1 *The order ρ_f and lower order λ_f of an entire function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r},$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, \ldots$ and $\log^{[0]} x = x$.

Definition 2 *The $p-$th order ρ_f^p and lower p-th order λ_f^p of an entire function f are defined as*

$$\rho_f^p = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f^p = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}.$$

Definition 3 *The $p-$th type σ_f^p of an entire function f is defined as*

$$\sigma_f^p = \limsup_{r\to\infty} \frac{\log^{[p-1]} M(r,f)}{r^{\rho_f^p}},\ 0 < \rho_f^p < \infty.$$

Since for $0 \leq r < R$, $\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f)$ $\{cf.[7]\}$ it is easy to see that

$$\rho_f^p = \limsup_{r\to\infty} \frac{\log^{[p+1]} \mu(r,f)}{\log r} \text{ and } \lambda_f^p = \liminf_{r\to\infty} \frac{\log^{[p+1]} \mu(r,f)}{\log r}\ .$$

In 1997 Lahiri and Banerjee [3] showed that how iteration can be made for any two entire functions f and g. According to them [3] the iteration of f with respect to g is defined as follows :

$$\begin{aligned}
f_1(z) &= f(z)\\
f_2(z) &= f(g(z)) = f(g_1(z))\\
f_3(z) &= f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))\\
&\quad \ldots\ldots\ldots \qquad \ldots\ldots\ldots \qquad \ldots\ldots\ldots\\
f_n(z) &= f(g(f.....(f(z) \text{ or } g(z))........)),\ \textit{according as } n \textit{ is odd or even},
\end{aligned}$$

and the iteration of g with respect to f is as follows :

$$\begin{aligned} g_1(z) &= g(z) \\ g_2(z) &= g(f(z)) = g(f_1(z)) \\ g_3(z) &= g(f(g(z))) = g(f_2(z)) = g(f(g_1(z))) \\ & \quad .... \quad .... \quad .... \\ g_n(z) &= g(f(g.....(g(z) \text{ or } f(z))........)), \textit{ according as } n \textit{ is odd or even.} \end{aligned}$$

Definition 4 *Let $a_1, a_2, ...a_k$ be linearly independent entire functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, ...a_k, f)$ the wronskian determinant of $a_1, a_2, ...a_k, f$ i.e.,*

$$L(f) = \begin{vmatrix} a_1 & a_2 & .. & .. & a_k & f \\ a_1' & a_2' & .. & .. & a_k' & f' \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ a_1^{(k)} & a_2^{(k)} & .. & .. & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

Definition 5 *Let 'a' be a complex number, finite or infinite. The Nevanlinna deficiency and Valiron deficiency of 'a' with respect to an entire function f are defined as*

$$\delta(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r\to\infty} \frac{m(r, a; f)}{T(r, f)}$$

and

$$\Delta(a; f) = 1 - \liminf_{r\to\infty} \frac{N(r, a; f)}{T(r, f)} = \limsup_{r\to\infty} \frac{m(r, a; f)}{T(r, f)}.$$

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C}\cup\{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a\neq\infty} \delta(a; f)+\delta(\infty; f) \leq 2$ (cf. $[[2], p.43]$). If in particular, $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

In the paper we would like to establish some new results based on the comparative growth properties related to the iteration of maximum terms of composite entire functions and wronskians generated by one of the factors. We do not explain the standard notations and definitions in the theory of entire functions as those are available in [8].

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([1]) *Let f and g be two entire functions. Then for all sufficiently large values of r,*

$$M(\frac{1}{8}M(\frac{r}{2},g)-|g(0)|,f)\leq M(r,f\circ g)\leq M(M(r,g),f)\ .$$

Lemma 2 ([2]) *Let f be an entire function and $0\leq r<R<\infty$. Then*

$$T\left(r,f\right)\leq\log^{+}M\left(r,f\right)\leq\frac{R+r}{R-r}T\left(R,f\right).$$

Lemma 3 ([6])*Let f and g be two entire functions. Then for all $r>0$,*

$$T\left(r,f\circ g\right)\geq\frac{1}{3}\log M\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o\left(1\right),f\right\}\ .$$

Lemma 4 ([4]) *Let f be a transcendental entire function having the maximum deficiency sum. Then*

$$\lim_{r\to\infty}\frac{T(r,L(f))}{T(r,f)}=1+k-k\delta(\infty;f).$$

Lemma 5 *Let f be a transcendental entire function with the maximum deficiency sum, then the p-th order and lower p-th order of $L(f)$ are same.*

Proof. *By Lemma 4, $\lim_{r\to\infty}\frac{\log^{[p]}T(r,L(f))}{\log^{[p]}T(r,f)}$ exists and is equal to 1.*
So

$$\begin{aligned}\rho_{L(f)}^{p} &= \limsup_{r\to\infty}\frac{\log^{[p]}T(r,f)}{\log r}.\lim_{r\to\infty}\frac{\log^{[p]}T(r,L(f))}{\log^{[p]}T(r,f)}\\ &= \rho_{f}^{p}.1=\rho_{f}^{p}.\end{aligned}$$

In a similar manner, $\lambda_{L(f)}^{p}=\lambda_{f.}^{p}$ ■

Lemma 6 *Let f and g be two entire functions both with non zero finite p-th order. Then for all sufficiently large values of r,*

$$\log^{[(n-1)p+1]} \mu(r, f_n) \leq \begin{Bmatrix} (\rho_f^p + \epsilon) \log M(r, g) + O(1) & \text{when } n \text{ is even} \\ (\rho_g^p + \epsilon) \log M(r, f) + O(1) & \text{when } n \text{ is odd.} \end{Bmatrix}$$

Proof. First let us consider n be even. Now in view of the second part of Lemma 1 and the inequality $\mu(r, f) \leq M(r, f)$ we obtain for all sufficiently large values of r,

$$\begin{aligned} \mu(r, f_n) &\leq M(r, f_n) \leq M(M(r, g_{n-1}), f) \\ i.e., \quad \log^{[p+1]} \mu(r, f_n) &\leq \log^{[p+1]} M(M(r, g_{n-1}), f) \\ &\leq (\rho_f^p + \epsilon) \log M(r, g_{n-1}). \end{aligned}$$

On calculation one can easily verify that for all sufficiently large values of r,

$$\log^{[(n-1)p+1]} \mu(r, f_n) \leq (\rho_f^p + \epsilon) \log M(r, g) + O(1).$$

Similarly it can be easily shown that for odd n

$$\log^{[(n-1)p+1]} \mu(r, f_n) \leq (\rho_g^p + \epsilon) \log M(r, f) + O(1).$$

This proves the lemma. ■

Lemma 7 *Let f and g be two entire functions both with non zero finite p-th order. Then for a sequence of values of r tending to infinity,*

$$\log^{[(n-1)p+1]} \mu(r, f_n) \leq \begin{Bmatrix} (\lambda_f^p + \epsilon) \log M(r, g) + O(1) & \text{when } n \text{ is even} \\ (\lambda_g^p + \epsilon) \log M(r, f) + O(1) & \text{when } n \text{ is odd.} \end{Bmatrix}$$

The proof of Lemma 7 is omited because it can be carried out in the line of Lemma 6.

Lemma 8 *Let f and g be two entire functions with non zero finite lower p-th order λ_f^p and λ_g^p respectively. Then for all sufficiently large values of r,*

$$\log^{[(n-1)p+1]} \mu(r, f_n) \geq \begin{Bmatrix} (\lambda_f^p - \epsilon) \log M(\frac{r}{2^n}, g) + O(1) & \text{when } n \text{ is even} \\ (\lambda_g^p - \epsilon) \log M(\frac{r}{2^n}, f) + O(1) & \text{when } n \text{ is odd.} \end{Bmatrix}$$

Proof. Putting $R = 2r$ in the inequality

$$\mu(r, f) \leq M(r, f) \leq \frac{R}{R-r}\mu(R, f)\{cf.[7]\}$$

we get that

$$\mu(r, f) \leq M(r, f) \leq 2\mu(2r, f). \tag{1}$$

Now we consider the case when n is even. Then using the second part of Lemma 1 and the second part of the inequality (1) we get for all sufficiently large values of r,

$$\begin{aligned}
\mu(r, f_n) &\geq \frac{1}{2}M(\frac{r}{2}, f_n) \\
&= \frac{1}{2}M(\frac{r}{2}, f(g_{n-1}) \\
&\geq \frac{1}{2}M(\frac{1}{32}M(\frac{r}{4}, g_{n-1}), f) \\
i.e., \quad \log^{[p+1]}\mu(r, f_n) &\geq (\lambda_f^p - \epsilon)\log M(\frac{r}{2^2}, g_{n-1}) + O(1) \\
i.e., \quad \log^{[p+2]}\mu(r, f_n) &\geq \log^{[2]} M(\frac{r}{2^2}, g_{n-1}) + O(1) \\
&\geq \log^{[2]} M(\frac{1}{16}M(\frac{r}{2^3}, f_{n-2}), g) + O(1).
\end{aligned}$$

Finaly on calculation we obtain that

$$\log^{[(n-1)p+1]}\mu(r, f_n) \geq (\lambda_f^p - \epsilon)\log M(\frac{r}{2^n}, g) + O(1).$$

Similarly for odd n it follows for all sufficiently large values of r,

$$\log^{[(n-1)p+1]}\mu(r, f_n) \geq (\lambda_g^p - \epsilon)\log M(\frac{r}{2^n}, f) + O(1).$$

This proves the lemma. ■

Lemma 9 *Let f and g be two entire functions with non zero finite p-th order ρ_f^p and ρ_g^p respectively. Then for a sequence of values of r tending to infinity,*

$$\log^{[(n-1)p+1]}\mu(r, f_n) \geq \begin{Bmatrix} (\rho_f^p - \epsilon)\log M(\frac{r}{2^n}, g) + O(1) & \text{when } n \text{ is even} \\ (\rho_g^p - \epsilon)\log M(\frac{r}{2^n}, f) + O(1) & \text{when } n \text{ is odd.} \end{Bmatrix}$$

The proof of Lemma 9 is omitted because it can be carried out in the line of Lemma 8.

Lemma 10 ([5]) *Let f be an entire function of finite lower order. If there exists entire functions a_i $(i = 1, 2, \cdots, n;\ n \leq \infty)$ satisfying $T(r, a_i) = o\{T(r, f)\}$ and $\sum_{i=1}^{n} \delta(a_i; f) = 1$, then $\lim_{r\to\infty} \frac{T(r,f)}{\log M(r,f)} = \frac{1}{\pi}$.*

3 Theorems.

In this section we present the main results of the paper.

Theorem 1 *Let f and g be two entire functions with (i) $0 < \lambda_f^p < \rho_f^p < \infty$, (ii) $0 < \lambda_g^p < \rho_g^p < \infty$, (iii) $0 < \rho_g^p < \lambda_f^p < \infty$, (iv) $0 < \sigma_g^p < \infty$ and (v) $\sum_{a\neq\infty} \delta(a; f) + \delta(\infty; f) = 2$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[(n-1)p+1]} \mu(r, f_n)}{\log^{[p]} \mu(r, L(f))} = 0$$

when n is even and $p \geq 1$.

Proof. Let us consider n to be even, then from Lemma 6 and the inequality (1) we obtain for all sufficiently large values of r

$$\begin{aligned} \log^{[(n-1)p+1]} \mu(r, f_n) &\leq (\rho_f^p + \epsilon) \log M(r, g) + O(1) \\ &\leq (\rho_f^p + \epsilon) \log \mu(2r, g) + O(1) \\ &\leq (\rho_f^p + \epsilon)(\sigma_g^p + \epsilon)(2r)^{(\rho_g^p+\epsilon)} + O(1). \end{aligned} \quad (2)$$

Now it is well known that for an entire function f, $T(r, f) \leq \log^+ M(r, f)$. So in view of inequality (1) and by Lemma 5 we get for all sufficiently large values of r

$$\log^{[p]} T(\frac{r}{2}, L(f)) \leq \log^{[p+1]} \mu(r, L(f)) + O(1)$$

$$i.e., \quad \log^{[p+1]} \mu(r, L(f)) \geq (\lambda_{L(f)}^p - \epsilon) \log(\frac{r}{2}) + O(1)$$

$$i.e., \quad \log^{[p]} \mu(r, L(f)) \geq r^{(\lambda_f^p - \epsilon)} + O(1). \quad (3)$$

Now from (2) and (3) we obtain for all sufficiently large values of r

$$\frac{\log^{[(n-1)p+1]}\mu(r,f_n)}{\log^{[p]}\mu(r,L(f))} \leq \frac{(\rho_f^p+\epsilon)(\sigma_g^p+\epsilon)(2r)^{(\rho_g^p+\epsilon)}+O(1)}{r^{(\lambda_f^p-\epsilon)}+O(1)}.$$

Since $\rho_g^p < \lambda_f^p$, we can choose $\epsilon(>0)$ in such a way that $\left(\rho_g^p+\epsilon\right) < (\lambda_f^p-\epsilon)$ and it follows from above that

$$\limsup_{r\to\infty}\frac{\log^{[(n-1)p+1]}\mu(r,f_n)}{\log^{[p]}\mu(r,L(f))} = 0.$$

This proves the theorem. ■

Remark 1 *The condition $\rho_g^p < \lambda_f^p$ in Theorem 1 is essential as we see in the following example.*

Example 1 *Let $f = \exp^{[p]} z$ and $g = \exp^{[p]}(z^2)$.*
Then $\lambda_f^p = \rho_f^p = 1$ and $\rho_g^p = \lambda_g^p = 2$ and $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f) = 2$.
Also $f_n = \exp^{[2n-2]p} z^2$ when n is even.
Taking $a_1 = \frac{1}{p}(p \geq 1)$, $a_2 = ... = a_k = 0$ in Definition 7 we get that

$$L(f) = \begin{vmatrix} a_1 & f \\ a_1' & f' \end{vmatrix} = \begin{vmatrix} \frac{1}{p} & \exp^{[p]} z \\ 0 & p\exp^{[p]} z \end{vmatrix} = \exp^{[p]} z = f.$$

In view of the inequality $\mu(r,f) \leq M(r,f) \leq 2\mu(2r,f)$ we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[(n-1)p+1]}\mu(r,f_n)}{\log^{[p]}\mu(r,L(f))} \leq \limsup_{r\to\infty}\frac{\log^{[(n-1)p+1]}M(r,f_n)}{\log^{[p]}M\{\frac{1}{2}(\frac{r}{2},L(f))\}}.$$

When n is even then

$$\log^{[(n-1)p+1]}M(r,f_n) = \log^{[(n-1)p+1]}M(r,\exp^{[2n-2]p} z^2) = \exp^{[np-p-1]} r^2.$$

Also

$$\log^{[p]}M\{\frac{1}{2}(\frac{r}{2},L(f))\} = \log^{[p]}M(\frac{r}{2},\exp^{[p]} z)+O(1) = \frac{r}{2}+O(1).$$

So

$$\limsup_{r\to\infty}\frac{\log^{[(n-1)p+1]}\mu(r,f_n)}{\log^{[p]}\mu(r,L(f))} = \limsup_{r\to\infty}\frac{\exp^{[np-p-1]} r^2}{\frac{r}{2}+O(1)} = \infty,$$

which is contrary to Theorem 1.

In the line of Theorem 1 we may state the following theorem without proof.

Theorem 2 *Let f and g be two entire functions with (i) $0 < \lambda_f^p < \rho_f^p < \infty$, (ii) $0 < \lambda_g^p < \rho_g^p < \infty$, (iii) $0 < \rho_f^p < \lambda_g^p < \infty$, (iv) $0 < \sigma_f^p < \infty$ and (v) $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$. Then*

$$\limsup_{r \to \infty} \frac{\log^{[(n-1)p+1]} \mu(r, f_n)}{\log^{[p]} \mu(r, L(g))} = 0$$

when n is odd and $p \geq 1$.

Theorem 3 *Let f and g be two entire functions with (i) $0 < \lambda_f^p < \rho_f^p < \infty$, (ii) $0 < \lambda_g^p < \rho_g^p < \infty$, (iii) $0 < \sigma_g^p < \infty$ and (iv) $\sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2$. Then*

$$\liminf_{r \to \infty} \frac{\log^{[(n-1)p+1]} \mu(r, f_n)}{\log^{[p]} \mu(r, L(g))} \leq 2^{\rho_g^p} . \rho_f^p . \sigma_g^p$$

when n is even and $p \geq 1$.

Proof. Let us consider n to be even, then from Lemma 6 and the inequality (1) we obtain for all sufficiently large values of r

$$\begin{aligned} \log^{[(n-1)p+1]} \mu(r, f_n) &\leq (\rho_f^p + \epsilon) \log M(r, g) + O(1) \\ &\leq (\rho_f^p + \epsilon) \log \mu(2r, g) + O(1) \\ &\leq (\rho_f^p + \epsilon)(\sigma_g^p + \epsilon)(2r)^{(\rho_g^p + \epsilon)} + O(1). \end{aligned} \tag{4}$$

Now it is well known that for an entire function f, $T(r, f) \leq \log^+ M(r, f)$. So in view of inequality (1) and by Lemma 5 we get for a sequence of values of r tending to infinity

$$\log^{[p]} T(\frac{r}{2}, L(g)) \leq \log^{[p+1]} \mu(r, L(g)) + O(1)$$

$$i.e., \quad \log^{[p+1]} \mu(r, L(g)) \geq (\rho_{L(g)}^p - \epsilon) \log(\frac{r}{2}) + O(1)$$

$$i.e., \quad \log^{[p]} \mu(r, L(g)) \geq r^{(\rho_g^p - \epsilon)} + O(1). \tag{5}$$

Now from (4) and (5) we obtain for a sequence of values of r

$$\frac{\log^{[(n-1)p+1]}\mu(r,f_n)}{\log^{[p]}\mu(r,L(g))} \leq \frac{(\rho_f^p+\epsilon)(\sigma_g^p+\epsilon)(2r)^{(\rho_g^p+\epsilon)}+O(1)}{r^{(\rho_g^p-\epsilon)}+O(1)}.$$

Since $\epsilon(>0)$ is arbitrary it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[(n-1)p+1]}\mu(r,f_n)}{\log^{[p]}\mu(r,L(g))} \leq 2^{\rho_g^p}.\rho_f^p.\sigma_g^p.$$

This proves the theorem. ■

In the line of Theorem 3 we may state the following theorem without proof.

Theorem 4 *Let f and g be two entire functions with (i) $0<\lambda_f^p<\rho_f^p<\infty$, (ii) $0<\lambda_g^p<\rho_g^p<\infty$, (iii) $0<\sigma_f^p<\infty$ and (iv) $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then*

$$\liminf_{r\to\infty}\frac{\log^{[(n-1)p+1]}\mu(r,f_n)}{\log^{[p]}\mu(r,L(f))} \leq 2^{\rho_f^p}.\rho_g^p.\sigma_f^p$$

when n is odd and $p\geq 1$.

Theorem 5 *Let f and g be two entire functions with (i) $0<\lambda_f^p<\rho_f^p<\infty$, (ii) $0<\lambda_g^p<\rho_g^p<\infty$, (iii) $0<\sigma_g^p<\infty$ and (iv) $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$. Also let there exist entire functions $b_i(i=1,2,...n;n\leq\infty)$ with $T(r,b_i)=o\{T(r,g)\}$ as $r\to\infty$ for $i=1,2,...,n$ and $\sum_{i=1}^{n}\delta(b_i;g)=1$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[(n-1)p+1]}\mu(r,f_n)}{\log^{[p]}\mu(r,L(f))T(r,L(g))}=0$$

when n is even and $p\geq 1$.

Proof. Let us consider n to be even, then from Lemma 6 and the inequality (1) we obtain for all sufficiently large values of r

$$\log^{[(n-1)p+1]}\mu(r,f_n)\leq(\rho_f^p+\epsilon)\log M(r,g)+O(1). \qquad (6)$$

Now it is well known that for an entire function f, $T(r,f) \leq \log^{+} M(r,f)$. So in view of inequality (1) and by Lemma 5 we get for all sufficiently large values of r

$$\log^{[p]} T(\frac{r}{2}, L(f)) \leq \log^{[p+1]} \mu(r, L(f)) + O(1)$$

$$i.e., \quad \log^{[p+1]} \mu(r, L(f)) \geq (\lambda^{p}_{L(f)} - \epsilon) \log(\frac{r}{2}) + O(1)$$

$$i.e., \quad \log^{[p]} \mu(r, L(f)) \geq r^{(\lambda^{p}_{f} - \epsilon)} + O(1). \tag{7}$$

Now from (6) and (7) we obtain for all sufficiently large values of r

$$\begin{aligned}\frac{\log^{[(n-1)p+1]} \mu(r, f_n)}{\log^{[p]} \mu(r, L(f)) T(r, L(g))} &\leq \frac{(\rho^{p}_{f} + \epsilon) \log M(r,g) + O(1)}{[r^{(\lambda^{p}_{f} - \epsilon)} + O(1)] T(r, L(g))} \\ &\leq \frac{(\rho^{p}_{f} + \epsilon)}{[r^{(\lambda^{p}_{f} - \epsilon)} + O(1)]} . \frac{\log M(r,g)}{T(r,g)} . \frac{T(r,g)}{T(r, L(g))}.\end{aligned}$$

Since $\epsilon (> 0)$ is arbitrary and by using Lemma 4 and Lemma 10 we get that

$$\begin{aligned}\limsup_{r\to\infty} \frac{\log^{[(n-1)p+1]} \mu(r, f_n)}{\log^{[p]} \mu(r, L(f)) T(r, L(g))} &\leq \limsup_{r\to\infty} \frac{\rho^{p}_{f}}{[r^{\lambda^{p}_{f}} + O(1)]} . \lim_{r\to\infty} \frac{\log M(r,g)}{T(r,g)} . \lim_{r\to\infty} \frac{T(r,g)}{T(r, L(g))} \\ &\leq 0.\pi . \frac{1}{1 + k - k\delta(\infty; f)} = 0.\end{aligned}$$

This proves the theorem. ■

In the line of Theorem 5 we may state the following theorem without proof.

Theorem 6 *Let f and g be two entire functions with (i) $0 < \lambda^{p}_{f} < \rho^{p}_{f} < \infty$, (ii) $0 < \lambda^{p}_{g} < \rho^{p}_{g} < \infty$, (iii) $0 < \sigma^{p}_{g} < \infty$ and (iv) $\sum\limits_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$. Also let there exist entire functions $b_i (i = 1, 2, ...n; n \leq \infty)$ with $T(r, b_i) = o\{T(r,g)\}$ as $r \to \infty$ for $i = 1, 2, ..., n$ and $\sum\limits_{i=1}^{n} \delta(b_i; g) = 1$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[(n-1)p+1]} \mu(r, f_n)}{\log^{[p]} \mu(r, L(g)) T(r, L(f))} = 0$$

when n is odd and $p \geq 1$.

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