Further Growth Estimations of Differential Monomials and Differential Polynomials in the Light of Zero Order and Weak Type

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Abstract

In this paper we investigate the comparative growth of composite entire or meromorphic functions and differential monomials , differential polynomials generated by one of the factors which improves some earlier results .

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1 Introduction, Definitions and Notations.

For any two transcendental entire functions f and g defined in the open complex plane $\mathbb{C}$, Clunie [4] proved that

$$\lim_{r \to \infty} \frac{T(r,fog)}{T(r,f)} = \infty \text{ and } \lim_{r \to \infty} \frac{T(r,fog)}{T(r,g)} = \infty \cdot$$

Singh [15] proved some comparative growth properties of $\log T(r, fog)$ and $T(r\,, f)$. He also raised the problem of investigating the comparative growth of $\log T(r, fog)$ and $T(r\,, g)$ which he was unable to solve . However , some results on the comparative growth of $\log T(r, fog)$ and $T(r\,, g)$ are proved in [11] .

Let f be a non-constant mereomorphic function defined in the open complex plane $\mathbb{C}$. Also let $n_{0j}, n_{1j}, \ldots. n_{kj}$ ($k \geq 1$) be non-negetive integers such that for each j , $\sum\limits_{i=0}^{k} n_{ij} \geq 1$. We call $M_j[f] = A_j\ (f)^{n_{0j}} \left(f^{(1)}\right)^{n_{1j}} \ldots. \left(f^{(k)}\right)^{n_{kj}}$ where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by f . The numbers $\gamma_{M_j} = \sum\limits_{i=0}^{k} n_{ij}$ and $\Gamma_{M_j} = \sum\limits_{i=0}^{k} (i+1)n_{ij}$ are called respectively the degree and weight of $M_j[f]$ {[8],[14]} . The expression $\mathrm{P}[f] = \sum_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f . The numbers $\gamma_P = \max_{1 \le j \le s} \gamma_{M_j}$ and $\Gamma_P = \max_{1 \le j \le s} \Gamma_{M_j}$ are called respectively the degree and weight of $\mathrm{P}[f]$ {[8] , [14]} . Also we call the numbers $\underline{\gamma_P} = \min_{1 \le j \le s} \gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of $\mathrm{P}[f]$ respectively . If $\underline{\gamma_P} = \gamma_P$, $\mathrm{P}[f]$ is called a homogeneous differential polynomial . In the paper we further investigate the question of Singh [15] mentioned earlier and prove some new results relating to the comparative growths of composite entire or meromorphic functions and differential monomials , differential polynomials generated by one of the factors . We do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in [18] and [9] . Throughout the paper we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f i.e. , for which $n_{0j} = 0$ for $j = 1, 2, \ldots s$. We consider only those $\mathrm{P}[f]$, $P_0[f]$ singularities of whose individual terms do not cancel each other . We also denote by $M[f]$ a differential monomial generated by a transcendental meromorphic function f.

The following definitions are well known .

Definition 1 *The order ρ_f and lower order λ_f of a meromorphic function f are defined as*

$$\rho_f = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log r} .$$

If f is entire , one can easily verify that

$$\rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log r} ,$$

where $\log^{[\boldsymbol{k}]}\boldsymbol{x} = \log(\log^{[\boldsymbol{k}-\mathbf{1}]}\boldsymbol{x})$ for $\boldsymbol{k} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots.$ and $\log^{[\mathbf{0}]}\boldsymbol{x} = \boldsymbol{x}$.

If $\rho_f < \infty$ then f is of finite order . Also $\rho_f = 0$ means that f is of order zero . In this connection Datta and Biswas [6] gave the following definition .

Definition 2 *[6] Let f be a meromorphic function of order zero. Then the quantities ρ_f^{**} and λ_f^{**} of f are defined by :*

$$\rho_f^{**} = \limsup_{r \to \infty} \frac{T(r,f)}{\log r} \quad \text{and} \quad \lambda_f^{**} = \liminf_{r \to \infty} \frac{T(r,f)}{\log r} .$$

If f is an entire function then clearly

$$\rho_f^{**} = \limsup_{r \to \infty} \frac{\log M(r,f)}{\log r} \quad \text{and} \quad \lambda_f^{**} = \liminf_{r \to \infty} \frac{\log M(r,f)}{\log r} .$$

Definition 3 *The type σ_f and lower type $\overline{\sigma}_f$ of a meromorphic function f are defined as*

$$\sigma_f = \limsup_{r \to \infty} \frac{T(r,f)}{r^{\rho_f}} \quad \text{and} \quad \overline{\sigma}_f = \liminf_{r \to \infty} \frac{T(r,f)}{r^{\rho_f}} , \quad 0 < \rho_f < \infty .$$

When f is entire, it can be easily verified that

$$\sigma_f = \limsup_{r \to \infty} \frac{\log M(r,f)}{r^{\rho_f}} \quad \text{and} \quad \overline{\sigma}_f = \liminf_{r \to \infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty .$$

Datta and Jha [5] gave the definition of weak type of a meromorphic function of finite positive lower order in the following way :

Definition 4 *[5] The weak type τ_f of a meromorphic function f of finite positive lower order λ_f is defined by*

$$\tau_f = \liminf_{r \to \infty} \frac{T(r,f)}{r^{\lambda_f}} .$$

For entire f,

$$\tau_f = \liminf_{r \to \infty} \frac{\log M(r,f)}{r^{\lambda_f}}, \; 0 < \lambda_f < \infty.$$

Similarly one can define the growth indicator $\overline{\tau_f}$ of a meromorphic function f of finite positive lower order λ_f as

$$\overline{\tau_f} = \limsup_{r \to \infty} \frac{T(r,f)}{r^{\lambda_f}} .$$

When f is entire, it can be easily verified that

$$\overline{\tau_f} = \limsup_{r \to \infty} \frac{\log M(r,f)}{r^{\lambda_f}}, \; 0 < \lambda_f < \infty.$$

Definition 5 *Let "a" be a complex number, finite or infinite. The Nevanlinna's deficiency and the Valiron deficiency of "a" with respect to a meromorphic function f are defined as*

$$\delta(a\,;f) = 1 - \limsup_{r \to \infty} \frac{N(r,a;f)}{T(r,f)} = \liminf_{r \to \infty} \frac{m(r,a;f)}{T(r,f)}$$

and

$$\Delta(a\,;f) = 1 - \liminf_{r \to \infty} \frac{N(r,a;f)}{T(r,f)} = \limsup_{r \to \infty} \frac{m(r,a;f)}{T(r,f)} .$$

Definition 6 *The quantity $\Theta(a\,;f)$ of a meromorphic function f is defined as follows*

$$\Theta(a\,;f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r,a;f)}{T(r,f)} .$$

Definition 7 *[17] For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $n(r,a;f \mid =1)$, the number of simple zeros of $f-a$ in $|z| \le r$. $N(r,a;f \mid =1)$ is defined in terms of $n(r,a;f \mid =1)$ in the usual way. We put*

$$\delta_1(a\,;f) = 1 - \limsup_{r \to \infty} \frac{N(r,a\,;f \mid =1)}{T(r,f)},$$

the deficiency of 'a' corresponding to the simple a-points of f i.e., simple zeros of $f-a$.

Yang [16] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta_1(a\,;f) > 0$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a\,;f) \le 4$.

Definition 8 *[12] For $a \in \mathbb{C} \cup \{\infty\}$, let $n_p(r,a;f)$ denotes the number of zeros of $f-a$ in $|z| \le r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\ge p$ is counted exactly p times ; and $N_p(r,a;f)$ is defined in terms of $n_p(r,a;f)$ in the usual way. We define*

$$\delta_p(a\,;f) = 1 - \limsup_{r\to\infty}\frac{N_p(r,a;f)}{T(r,f)} .$$

Definition 9 *[3] $P[f]$ is said to be admissible if*

(i) $P[f]$ is homogeneous , or

(ii) $P[f]$ is non homogeneous and $m(r,f) = S(r,f)$.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel .

Lemma 1 *[1] If f is meromorphic and g is entire then for all sufficiently large values of r,*

$$T(r,fog) \le \{1+o(1)\}\frac{T(r,g)}{\log M(r,g)}T(M(r,g),f) .$$

Lemma 2 *[2] Let f be meromorphic and g be entire and suppose that $0 < \mu < \rho_g \le \infty$. Then for a sequence of values of r tending to infinity ,*

$$T(r,fog) \ge T(\exp(r^{\mu}),f) .$$

Lemma 3 *[10] Let f be meromorphic and g be entire such that $0 < \mu < \rho_g \le \infty$ and $\lambda_f > 0$. Then for a sequence of values of r tending to infinity ,*

$$T(r,fog) > T(\exp(r^{\mu}),g) .$$

Lemma 4 *[7] Let f be a meromorphic function and g be an entire function such that $\lambda_g < \mu < \infty$ and $0 < \lambda_f \le \rho_f < \infty$. Then for a sequence of values of r tending to infinity ,*

$$T(r,fog) < T(\exp(r^{\mu}),f) .$$

Lemma 5 *[7] Let f be a meromorphic function of finite order and g be an entire function with $0 < \lambda_g < \mu < \infty$. Then for a sequence of values of r tending to infinity ,*

$$T(r,fog) < T(\exp(r^{\mu}),g) .$$

Lemma 6 *[3] Let $P_0[f]$ be admissible . If f is of finite order or of non-zero lower order and $\sum_{a \ne \infty} \Theta(a\,;f) = 2$ then*

$$\lim_{r\to\infty}\frac{T(r,P_0[f])}{T(r,f)} = \Gamma_{P_0[f]} .$$

Lemma 7 *[3] Let f be either of finite order or of non-zero lower order such that $\Theta(\infty\,;f) =$*

$\sum_{a \neq \infty} \delta_p(a\,;f) = 1$ *or* $\delta(\infty\,;f) = \sum_{a \neq \infty} \delta(a\,;f) = 1$. *Then for homogeneous* $P_0[f]$,

$$\lim_{r \to \infty} \frac{T(r\,, P_0[\,f\,])}{T(r,f)} = \gamma_{P_0[\,f\,]}\,.$$

Lemma 8 *Let* f *be a meromorphic function of finite order or of non zero lower order . If* $\sum_{a \neq \infty} \Theta(a\,;f) = 2$ *, then the order (lower order) of homogeneous* $P_0[f]$ *is same as that of* f*. Also* $\sigma_{P_0[\,f\,]}, \overline{\sigma}_{P_0[\,f\,]}, \tau_{P_0[\,f\,]}$ *and* $\overline{\tau}_{P_0[\,f\,]}$ *are* $\Gamma_{P_0[\,f\,]}$ *times that of* f *if* f *is of positive finite order .*

Proof. By Lemma 6 , $\lim_{r \to \infty} \frac{\log T(r\,, P_0[\,f\,])}{\log T(r,f)}$ exists and is equal to 1.

$$\rho_{P_0[\,f\,]} = \limsup_{r \to \infty} \frac{\log T(r\,, P_0[\,f\,])}{\log r}$$

$$= \limsup_{r \to \infty} \frac{\log T(r,f))}{\log r} \cdot \lim_{r \to \infty} \frac{\log T(r\,, P_0[\,f\,])}{\log T(r,f)}$$

$$= \rho_f . 1 = \rho_f .$$

In a similar manner , $\lambda_{P_0[\,f\,]} = \lambda_f$.

Again by Lemma 6,

$$\sigma_{P_0[\,f\,]} = \limsup_{r \to \infty} \frac{T(r\,, P_0[\,f\,])}{r^{\rho_{P_0[\,f\,]}}}$$

$$= \lim_{r \to \infty} \frac{T(r\,, P_0[\,f\,])}{T(r,f)} \cdot \limsup_{r \to \infty} \frac{T(r,f)}{r^{\rho_f}} = \Gamma_{P_0[\,f\,]} . \sigma_f \,.$$

Similarly $\overline{\sigma}_{P_0[\,f\,]} = \Gamma_{P_0} . \overline{\sigma}_f$.

Also

$$\tau_{P_0[\,f\,]} = \liminf_{r \to \infty} \frac{T(r\,, P_0[\,f\,])}{r^{\lambda_{P_0[\,f\,]}}}$$

$$= \lim_{r \to \infty} \frac{T(r\,, P_0[\,f\,])}{T(r,f)} \cdot \liminf_{r \to \infty} \frac{T(r,f)}{r^{\lambda_f}} = \Gamma_{P_0} . \tau_f \,.$$

Analogously $\overline{\tau}_{P_0[\,f\,]} = \Gamma_{P_0[\,f\,]} . \overline{\tau}_f$.

This proves the lemma .

Lemma 9 *Let* f *be a meromorphic function of finite order or of non zero lower order such that* $\Theta(\infty\,;f) = \sum_{a \neq \infty} \delta_p(a\,;f) = 1$. *Then the order (lower order) of homogeneous* $P_0[f]$ *and* f *are same . Also* $\sigma_{P_0[\,f\,]}, \overline{\sigma}_{P_0[\,f\,]}, \tau_{P_0[\,f\,]}$ *and* $\overline{\tau}_{P_0[\,f\,]}$ *are* $\gamma_{P_0[\,f\,]}$ *times that of* f *when* f *is of finite positive order.*

We omit the proof of the lemma because it can be carried out in the line of Lemma 8 and with the help of Lemma 7 .

In a similar manner we can state the following lemma without proof.

Lemma 10 *Let f be a meromorphic function of finite order or of non- zero lower order such that $\delta(\infty\,; f) = \sum_{a \neq \infty} \delta(a\,; f) = 1$. Then for every homogeneous $P_0[f]$ the order (lower order) of $P_0[f]$ is same as that of f . Also the $\sigma_{P_0[f]}, \overline{\sigma}_{P_0[f]}, \tau_{P_0[f]}$ and $\overline{\tau}_{P_0[f]}$ are $\gamma_{P_0[f]}$ times that of f when f is of finite positive order.*

Lemma 11 *[13] Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a\,; f) \leq 4$, then*

$$\lim_{r \to \infty} \frac{T(r,M[f])}{T(r,f)} = \Gamma_M - (\Gamma_M - \gamma_M)\,\Theta(\infty\,; f)\,,$$

where

$$\Theta(\infty\,; f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r,f)}{T(r,f)}\;.$$

Lemma 12 *If f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a\,; f) \leq 4$, then the order and lower order of $M[f\,]$ are same as those of f . Also $\sigma_{M[f]}, \overline{\sigma}_{M[f]}, \tau_{M[f]}$ and $\overline{\tau}_{M[f]}$ are $\{\Gamma_M - (\Gamma_M - \gamma_M)\,\Theta(\infty\,; f)\}$ times that of f when f is of finite positive order.*

We omit the proof of the lemma because it can be carried out in the line of Lemma 8 and with the help of Lemma 11 .

3 Theorems.

In this section we present the main results of the paper .

Theorem 1 *Let f be a meromorphic function and g be an entire function such that (i) $0 < \lambda_f \leq \rho_f < \infty$, (ii) $\lambda_f = \lambda_g$, (iii) $\tau_f > 0$, (iv) $\overline{\tau}_g < \infty$ and (v) $\lambda_f < \rho_g \leq \infty$. Also let $\sum_{a \neq \infty} \Theta(a\,; f) = 2$. Then*

$$\frac{\max\{\lambda_f, \lambda_g\}}{\Gamma_{P_0[f]}\,.\overline{\tau}_f} \leq \limsup_{r \to \infty} \frac{\log T(r, fog)}{T(r\,, P_0[f])} \leq \rho_f\, \frac{\overline{\tau}_g}{\Gamma_{P_0[f]}\;\tau_f}\,.$$

Proof. Let us suppose that $0 < \varepsilon < \min\{\lambda_f\,, \Gamma_{P_0[f]}\;\tau_f\}$.

Since $\lambda_f < \rho_g$, in view of Lemma 2 we obtain for a sequence of values of r tending to infinity that

$$\log T(r, fog) \geq \log T(\exp(r^{\lambda_f}), f)$$

$$i.e., \log T(r, fog) \geq (\lambda_f - \varepsilon)\log\exp(r^{\lambda_f})$$

$$i.e., \log T(r, fog) \geq (\lambda_f - \varepsilon)\, r^{\lambda_f}\,. \tag{1}$$

Again by Lemma 8 , we have for all sufficiently large values of r,

$$T(r, P_0[\,f\,]) \leq (\overline{\tau}_{P_0[\,f\,]} + \varepsilon)\, r^{\lambda_{P_0[\,f\,]}}$$

$$i.e.,\ T(r, P_0[\,f\,]) \leq (\Gamma_{P_0[\,f\,]}\, \overline{\tau}_f + \varepsilon)\, r^{\lambda_f}.$$

(2)

Therefore from (1) and (2) it follows for a sequence of values of r tending to infinity that

$$\frac{\log T(r, fog)}{T(r, P_0[\,f\,])} \geq \frac{(\lambda_f - \varepsilon)\, r^{\lambda_f}}{(\Gamma_{P_0[\,f\,]}\, \overline{\tau}_f + \varepsilon)\, r^{\lambda_f}}$$

$$i.e.,\ \limsup_{r \to \infty} \frac{\log T(r, fog)}{T(r, P_0[\,f\,])} \geq \frac{\lambda_f}{\Gamma_{P_0[\,f\,]}\, \overline{\tau}_f}. \qquad (3)$$

Similarly in view of Lemma 3 we get that

$$\limsup_{r \to \infty} \frac{\log T(r, fog)}{T(r, P_0[\,f\,])} \geq \frac{\lambda_g}{\Gamma_{P_0[\,f\,]}\, \overline{\tau}_f}. \qquad (4)$$

Again we have from Lemma 1 for all sufficiently large values of r ,

$$T(r, fog) \leq \{1 + o(1)\}\, T(M(r, g), f)$$

$$i.e.,\ \log T(r, fog) \leq \left(\rho_f + \varepsilon\right) \log M(r, g) + O(1)$$

$$i.e.,\ \liminf_{r \to \infty} \frac{\log T(r, fog)}{T(r, P_0[\,f\,])} \leq \left(\rho_f + \varepsilon\right) \liminf_{r \to \infty} \frac{\log M(r, g)}{T(r, P_0[\,f\,])}.$$

(5)

Also for all sufficiently large values of r

$$\log M(r, g) \leq (\overline{\tau}_g + \varepsilon)\, r^{\lambda_g}. \qquad (6)$$

Again in view of Lemma 8 we obtain for a sequence of values of r tending to infinity that

$$T(r, P_0[\,f\,] \geq (\tau_{P_0[\,f\,]} - \varepsilon)\, r^{\lambda_{P_0[\,f\,]}}$$

$$i.e.,\ T(r, P_0[\,f\,] \geq (\Gamma_{P_0[\,f\,]}\, \tau_f - \varepsilon)\, r^{\lambda_f}.$$

(7)

Since $\lambda_f = \lambda_g$ we get from (6) and (7) for a sequence of values of r tending to infinity that

$$\liminf_{r \to \infty} \frac{\log M(r, g)}{T(r, P_0[\,f\,])} \leq \frac{\overline{\tau}_g}{\Gamma_{P_0}\, \tau_f}. \qquad (8)$$

Since $\varepsilon(> 0)$ is arbitrary , from (6) and (8) we obtain that

$$\liminf_{r \to \infty} \frac{\log T(r, fog)}{T(r, P_0[\,f\,])} \leq \rho_f\, \frac{\overline{\tau}_g}{\Gamma_{P_0[\,f\,]}\, \tau_f}. \qquad (9)$$

Thus the theorem follows from (3) , (4) and (9) .

Remark 1 *If we take* “ $\Theta(\infty;f) = \sum_{a \neq \infty} \delta_p(a;f) = 1$ *or* $\delta(\infty;f) = \sum_{a \neq \infty} \delta(a;f) = 1$” *instead of* “ $\sum_{a \neq \infty} \Theta(a;f) = 2$” *in Theorem* 1 *and the other conditions remain the same then one can easily prove that*

$$\frac{\max\{\lambda_f, \lambda_g\}}{\gamma_{P_0[f]}.\overline{\tau}_f} \leq \limsup_{r \to \infty} \frac{\log T(r, fog)}{T(r, P_0[f])} \leq \rho_f \frac{\overline{\tau}_g}{\gamma_{P_0[f]}\ \tau_f} .$$

In the line of Theorem 1 and with the help of Lemma 12 we may state the following theorem without proof.

Theorem 2 *Let* f *be a transcendental meromorphic function and* g *be an entire function such that* (*i*) $0 < \lambda_f \leq \rho_f < \infty$, (*ii*) $\lambda_f = \lambda_g$, (*iii*) $\tau_f > 0$, (*iv*) $\overline{\tau}_g < \infty$ *and* (*v*) $\lambda_f < \rho_g$. *Also let* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;f) \leq 4$. *Then*

$$\frac{\max\{\lambda_f, \lambda_g\}}{\Gamma_M - (\Gamma_M - \gamma_M)\,\Theta(\infty;f).\overline{\tau}_f} \leq \limsup_{r \to \infty} \frac{\log T(r, fog)}{T(r, M[f])}$$

$$\leq \rho_f \frac{\overline{\tau}_g}{\Gamma_M - (\Gamma_M - \gamma_M)\,\Theta(\infty;f)\,\tau_f} .$$

In the line of Theorem 1 we may also state the following theorem without proof.

Theorem 3 *Let* f *be a meromorphic function and* g *be an entire function with* (*i*) $0 < \lambda_f \leq \rho_f < \infty$, (*ii*) $0 < \lambda_g < \rho_g < \infty$, (*iii*) $0 < \overline{\sigma}_g \leq \sigma_g < \infty$, *and* (*iv*) $0 < \overline{\tau}_g \leq \tau_g < \infty$. *Also let*

$\Theta(\infty;g) = \sum_{a \neq \infty} \delta_p(a;g) = 1$ *or* $\delta(\infty;g) = \sum_{a \neq \infty} \delta(a;g) = 1$. *Then*

$$\frac{\max\{\lambda_f, \lambda_g\}}{\gamma_{P_0[g]}.\overline{\tau}_g} \leq \limsup_{r \to \infty} \frac{\log T(r, fog)}{T(r, P_0[g])} \leq \frac{\rho_f}{\gamma_{P_0[g]}} \min\left\{\frac{\sigma_g}{\overline{\sigma}_g}, \frac{\overline{\tau}_g}{\tau_g}\right\} .$$

Remark 2 *In addition to the conditions of Theorem* 3 *if* f *be a meromorphic function with* $0 < \lambda_f^{**} \leq \rho_f^{**} < \infty$ *then by Definition* 2 *and similar process of Theorem* 1 *one can verify that*

$$\frac{\lambda_f^{**}}{\gamma_{P_0[g]}.\overline{\tau}_g} \leq \limsup_{r \to \infty} \frac{T(r, fog)}{T(r, P_0[g])} \leq \frac{\{1+o(1)\}\,\rho_f^{**}}{\gamma_{P_0[g]}} \min\left\{\frac{\sigma_g}{\overline{\sigma}_g}, \frac{\overline{\tau}_g}{\tau_g}\right\} .$$

Remark 3 *Under the same condition of Theorem* 3 , *if we take* “ $\sum_{a \neq \infty} \Theta(a;f) = 2$ ” *instead of* “ $\Theta(\infty;g) = \sum_{a \neq \infty} \delta_p(a;g) = 1$ *or* $\delta(\infty;g) = \sum_{a \neq \infty} \delta(a;g) = 1$ ” , *then the following result holds :*

$$\frac{1}{\Gamma_{P_0[g]}.\overline{\tau}_g} \max\{\lambda_f, \lambda_g\} \leq \limsup_{r \to \infty} \frac{\log T(r, fog)}{T(r, P_0[g])} \leq \frac{\rho_f}{\Gamma_{P_0[g]}} \min\left\{\frac{\sigma_g}{\overline{\sigma}_g}, \frac{\overline{\tau}_g}{\tau_g}\right\} .$$

Remark 4 *In Remark* 2 *if we take* $0 < \lambda_f^{**} \leq \rho_f^{**} < \infty$ *instead of* $0 < \lambda_f \leq \rho_f < \infty$ *and the other conditions remain the same then it can be shown that*

$$\frac{\lambda_f^{**}}{\Gamma_{P_0[g]}.\overline{\tau}_g} \leq \limsup_{r\to\infty} \frac{T(r,fog)}{T(r,P_0[g])} \leq \frac{\{1+o(1)\}\,\rho_f^{**}}{\Gamma_{P_0[g]}} \min\left\{\frac{\sigma_g}{\overline{\sigma}_g}, \frac{\overline{\tau}_g}{\tau_g}\right\}.$$

Theorem 4 *Let* f *be a meromorphic function and* g *be a transcendental entire function such that* (i) $0 < \lambda_f \leq \rho_f < \infty$, (ii) $0 < \lambda_g \leq \rho_g < \infty$, (iii) $0 < \overline{\sigma}_g \leq \sigma_g < \infty$, (iv) $0 < \overline{\tau}_g \leq \tau_g < \infty$. *Also let* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) \leq 4$. *Then*

$$\frac{\max\{\lambda_f,\lambda_g\}}{\{\Gamma_M-(\Gamma_M-\gamma_M)\,\Theta(\infty;g)\}.\overline{\tau}_g} \leq \limsup_{r\to\infty} \frac{\log T(r,fog)}{T(r,M[g])} \leq$$

$$\frac{\rho_f \min\left\{\frac{\sigma_g}{\overline{\sigma}_g}, \frac{\overline{\tau}_g}{\tau_g}\right\}}{\{\Gamma_M-(\Gamma_M-\gamma_M)\,\Theta(\infty;g)\}}.$$

The proof is omitted because it can be carried out in the line of Theorem 3 and with the help of Lemma 12 .

Remark 5 *Under the same conditions of Theorem* 4 *if* f *be a meromorphic function with order zero and* $0 < \lambda_f^{**} \leq \rho_f^{**} < \infty$ *then with the help of Definition* 2 *and similar process of Theprem* 4 *one can easily verify that*

$$\frac{\lambda_f^{**}}{\{\Gamma_M-(\Gamma_M-\gamma_M)\,\Theta(\infty;g)\}.\overline{\tau}_g} \leq \limsup_{r\to\infty} \frac{\log T(r,fog)}{T(r,M[g])} \leq$$

$$\frac{\{1+o(1)\}\,\rho_f^{**} \min\left\{\frac{\sigma_g}{\overline{\sigma}_g}, \frac{\overline{\tau}_g}{\tau_g}\right\}}{\{\Gamma_M-(\Gamma_M-\gamma_M)\,\Theta(\infty;g)\}}.$$

Theorem 5 *Let* f *be a meromorphic function and* g *be an entire function such that* (i) $0 < \lambda_f \leq \rho_f < \infty$, (ii) $\rho_f = \rho_g$, (iii) $\sigma_g < \infty$, (iv) $\overline{\sigma}_f > 0$ *and* $\Theta(\infty;f) = \sum_{a\neq\infty} \delta_p(a;f) = 1$ *or* $\delta(\infty;f) = \sum_{a\neq\infty} \delta(a;f) = 1$. *Then*

$$\liminf_{r\to\infty} \frac{\log T(r,fog)}{T(r,P_0[f])} \leq \frac{1}{\gamma_{P_0[f]}} \min\left\{\rho_f \frac{\sigma_g}{\sigma_f}, \rho_f \frac{\overline{\sigma}_g}{\overline{\sigma}_f}, \lambda_f \frac{\sigma_g}{\overline{\sigma}_f}\right\}.$$

Proof. As $\boldsymbol{T(r,g)} \leq \log^+ \boldsymbol{M(r,g)}$, we have from Lemma 1 for a sequence of values of r tending to infinity that

$$T(r,fog) \leq \{1+o(1)\}\,T(M(r,g),f)$$

$$i.e., \log T(r,fog) \leq (\lambda_f+\varepsilon)\log M(r,g) + O(1)$$

$$i.e., \liminf_{r\to\infty} \frac{\log T(r,fog)}{T(r,P_0[f])} \leq (\lambda_f+\varepsilon) \liminf_{r\to\infty} \frac{\log M(r,g)}{T(r,P_0[f])}. \tag{10}$$

Now from the definition of type it follows for all sufficiently large values of r

$$\log M(r,g) \leq (\sigma_g+\varepsilon)\, r^{\rho_g}. \tag{11}$$

Also from the definition of lower type we obtain for a sequence of values of r tending to infinity that

$$\log M(r,g) \leq (\overline{\sigma}_g + \varepsilon)\, r^{\rho_g}\ . \tag{12}$$

Again by Lemma 9 and Lemma 10 , we have for all sufficiently large values of r that

$$T(r, P_0[f]) \geq (\overline{\sigma}_{P_0[f]} - \varepsilon)\, r^{\rho_{P_0[f]}}$$

$$i.e.,\ T(r, P_0[f]) \geq (\gamma_{P_0[f]}\, \overline{\sigma}_f - \varepsilon)\, r^{\rho_f}. \tag{13}$$

Similarly with the help of Lemma 9 and Lemma 10 we obtain for a sequence of values of r tending to infinity that

$$T(r, P_0[f]) \leq (\sigma_{P_0[f]} - \varepsilon)\, r^{\rho_{P_0[f]}}$$

$$i.e.,\ T(r, P_0[f]) \leq (\gamma_{P_0[f]}\, \sigma_f - \varepsilon)\, r^{\rho_f}. \tag{14}$$

Since $\rho_f = \rho_g$ we get from (11) and (14) for a sequence of values of r tending to infinity that

$$\liminf_{r \to \infty} \frac{\log M(r,g)}{T(r, P_0[f])} \leq \frac{\sigma_g}{\gamma_{P_0[f]}\, \sigma_f}\ . \tag{15}$$

Similarly from (12) and (13) it follows for a sequence of values of r tending to infinity that

$$\liminf_{r \to \infty} \frac{\log M(r,g)}{T(r, P_0[f])} \leq \frac{\overline{\sigma}_g}{\gamma_{P_0[f]}\, \overline{\sigma}_f}\ . \tag{16}$$

Also we obtain from (11) and (13) for all sufficiently large values of r,

$$\liminf_{r \to \infty} \frac{\log M(r,g)}{T(r, P_0[f])} \leq \frac{\sigma_g}{\gamma_{P_0[f]}\, \overline{\sigma}_f}\ . \tag{17}$$

Since $\varepsilon\ (> 0)$ is arbitrary , from (5) and (15) we obtain that

$$\liminf_{r \to \infty} \frac{\log T(r, fog)}{T(r, P_0[f])} \leq \rho_f \frac{\sigma_g}{\gamma_{P_0[f]}\, \sigma_f}\ . \tag{18}$$

Similarly from (5) and (16) it follows that

$$\liminf_{r \to \infty} \frac{\log T(r, fog)}{T(r, P_0[f])} \leq \rho_f \frac{\overline{\sigma}_g}{\gamma_{P_0[f]}\, \overline{\sigma}_f}\ . \tag{19}$$

Also we get from (10) and (17) that

$$\liminf_{r \to \infty} \frac{\log T(r, fog)}{T(r, P_0[f])} \leq \lambda_f \frac{\sigma_g}{\gamma_{P_0[f]}\, \overline{\sigma}_f}\ . \tag{20}$$

Thus the theorem follows from (18) , (19) and (20) .

Remark 6 *Theorem* 5 *remains true with* $\Gamma_{P_0[f]}$ *instead of* $\gamma_{P_0[f]}$ *if we replace the condition* $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ *or* $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ *by* $\sum_{a \neq \infty} \Theta(a; f) = 2$ *and the other conditions remain the same .*

Theorem 6 *Let* f *be a transcendental meromorphic function and* g *be an entire function such that* (i) $0 < \lambda_f \leq \rho_f < \infty$, (ii) $\rho_f = \rho_g$, (iii) $\sigma_g < \infty$, (iv) $\overline{\sigma}_f > 0$ *and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a\,; f) \leq 4$. *Then*

$$\liminf_{r \to \infty} \frac{\log T(r, fog)}{T(r, M[f])} \leq \frac{1}{\{\Gamma_M - (\Gamma_M - \gamma_M)\ \Theta(\infty\,; f)\}} \min\left\{ \rho_f \frac{\sigma_g}{\sigma_f}, \rho_f \frac{\overline{\sigma}_g}{\overline{\sigma}_f}, \lambda_f \frac{\sigma_g}{\overline{\sigma}_f} \right\}.$$

The proof of the theorem can be established in the line of Theorem 6 and with the help of Lemma 12 and therefore is omitted.

In the line of Theorem 5 we may state the following theorem without proof.

Theorem 7 *Let* f *be a meromorphic function and* g *be an entire function such that* (i) $0 < \lambda_f < \infty$, (ii) $\sigma_g < \infty$, (iii) $\overline{\sigma}_f > 0$, *and* (iv) $\sum_{a \neq \infty} \Theta(a\,; g) = 2$. *Then*

$$\liminf_{r \to \infty} \frac{\log T(r, fog)}{T(r, P_0[g])} \leq \frac{\lambda_f}{\Gamma_{P_0[g]}} \cdot \frac{\sigma_g}{\overline{\sigma}_g}.$$

Remark 7 *In addition to the conditions of Theorem 7 if* f *be a meromorphic function with* $0 < \lambda_f^{**} < \infty$ *then one can easily verify that*

$$\liminf_{r \to \infty} \frac{T(r, fog)}{T(r, P_0[g])} \leq \frac{\{1 + o(1)\}\, \lambda_f^{**}}{\Gamma_{P_0[g]}} \cdot \frac{\sigma_g}{\overline{\sigma}_g}.$$

Remark 8 *Theorem 7 and Remark 7 remain true with* $\gamma_{P_0[g]}$ *instead of* $\Gamma_{P_0[g]}$ *if we replace the condition* $\sum_{a \neq \infty} \Theta(a\,; g) = 2$ *by* $\Theta(\infty\,; g) = \sum_{a \neq \infty} \delta_p(a\,; g) = 1$ *or* $\delta(\infty\,; g) = \sum_{a \neq \infty} \delta(a\,; g) = 1$ *and the other conditions are same.*

In the line of Theorem 7 and in view of Lemma 12 we may state the following theorem without proof.

Theorem 8 *Let* f *be a meromorphic function and* g *be a transcendental entire function such that* (i) $0 < \lambda_f < \infty$ (ii) $\sigma_g < \infty$, (iii) $\overline{\sigma}_f > 0$ *and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a\,; f) \leq 4$. *Then*

$$\liminf_{r \to \infty} \frac{\log T(r, fog)}{T(r, M[g])} \leq \frac{\lambda_f}{\{\Gamma_M - (\Gamma_M - \gamma_M)\ \Theta(\infty\,; g)\}} \cdot \frac{\sigma_g}{\overline{\sigma}_g}.$$

Remark 9 *In addition the conditions of Theorem 8 if* f *be a meromorphic function with* $0 < \lambda_f^{**} < \infty$ *then one can easily verify that*

$$\liminf_{r \to \infty} \frac{T(r, fog)}{T(r, M[g])} \leq \frac{\{1 + o(1)\}\, \lambda_f^{**}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\ \Theta(\infty\,; g)\}} \cdot \frac{\sigma_g}{\overline{\sigma}_g}.$$

Theorem 9 *Let f be a meromorphic function and g be an entire function such that* $0 < \lambda_g < \rho_f$, $0 < \lambda_f \leq \rho_f < \infty$, $\overline{\sigma}_f > 0$ *and* $\sum_{a \neq \infty} \Theta(a\,;f) = 2$ *. Then*

$$\liminf_{r \to \infty} \frac{\log T(r,fog)}{T(r\,,P_0[\,f\,])} \leq \min \frac{1}{\Gamma_{P_0[\,f\,]}} \left\{ \frac{\rho_f}{\overline{\sigma}_f}, \frac{\rho_g}{\overline{\sigma}_f} \right\}.$$

Proof. Since $\lambda_g < \rho_f$, in view of Lemma 4 we obtain for a sequence of values of r tending to infinity that

$$\log T(r,fog) < \log T\{\exp(r^{\rho_f}), f\}$$

$$i.e., \log T(r,fog) < (\rho_f + \varepsilon) \log \exp(r^{\rho_f})$$

$$i.e., \log T(r,fog) < (\rho_f + \varepsilon)\, r^{\rho_f}\,. \tag{21}$$

Again by Lemma 8 , we have for all sufficiently large values of r,

$$T(r, P_0[\,f\,]) \geq (\overline{\sigma}_{P_0[\,f\,]} - \varepsilon)\, r^{\rho_{P_0[\,f\,]}}$$

$$i.e.,\, T(r, P_0[\,f\,]) \geq (\Gamma_{P_0[\,f\,]}\,\overline{\sigma}_f - \varepsilon)\, r^{\rho_f}. \tag{22}$$

Therefore from (21) and (22) it follows for a sequence of values of r tending to infinity

$$\frac{\log T(r,fog)}{T(r,P_0[\,f\,])} \leq \frac{(\rho_f + \varepsilon)\; r^{\rho_f}}{(\Gamma_{P_0[\,f\,]}\,\overline{\sigma}_f - \varepsilon)\, r^{\rho_f}}$$

$$i.e., \limsup_{r \to \infty} \frac{\log T(r,fog)}{T(r\,,P_0[\,f\,])} \leq \frac{\rho_f}{\Gamma_{P_0[\,f\,]}\overline{\sigma}_f}\,. \tag{23}$$

Similarly in view of Lemma 5 we get that

$$\limsup_{r \to \infty} \frac{\log T(r,fog)}{T(r\,,P_0[\,f\,])} \leq \frac{\rho_g}{\Gamma_{P_0[\,f\,]}\overline{\sigma}_f}\,. \tag{24}$$

Thus the theorem follows from (23) and (24) .

Remark 10 *Theorem 9 remains true with* $\gamma_{P_0[\,f\,]}$ *instead of* $\Gamma_{P_0[\,f\,]}$ *if we replace the condition* $\sum_{a \neq \infty} \Theta(a\,;f) = 2$ *by* $\Theta(\infty\,;f) = \sum_{a \neq \infty} \delta_p(a\,;f) = 1$ *or* $\delta(\infty\,;f) = \sum_{a \neq \infty} \delta(a\,;f) = 1$ *and the other conditions remain the same .*

Theorem 10 *Let f be a meromorphic function and g be an entire function such that* $0 < \lambda_g < \rho_f$, $0 < \lambda_f \leq \rho_f < \infty$, $\overline{\sigma}_g > 0$ *and* $\Theta(\infty\,;g) = \sum_{a \neq \infty} \delta_p(a\,;g) = 1$ *or* $\delta(\infty\,;g) = \sum_{a \neq \infty} \delta(a\,;g) = 1$ *. Then*

$$\liminf_{r \to \infty} \frac{\log T(r,fog)}{T(r\,,P_0[g])} \leq \min \frac{1}{\gamma_{P_0[\,f\,]}} \left\{ \frac{\rho_f}{\overline{\sigma}_g}, \frac{\rho_g}{\overline{\sigma}_g} \right\}.$$

Theorem 10 can be carried out in the line of Theorem 9 and therefore its proof is omitted .

Remark 11 *if we take* $\sum_{a \neq \infty} \Theta(a\,;g) = 2$ *instead of* $\Theta(\infty\,;g) = \sum_{a \neq \infty} \delta_p(a\,;g) = 1$ *or* $\delta(\infty\,;g) = \sum_{a \neq \infty} \delta(a\,;g) = 1$ *in Theorem* 10 *and the other conditions remain the same then Theorem* 10 *remains valid with* $\Gamma_{P_0[\,g\,]}$ *instead of* $\gamma_{P_0[\,g\,]}$.

The following two theorems can be carried out in view of Lemma 14 and in the similar way of Theorem 9 and Theorem 10 respectively . Hence the proof is omitted .

Theorem 11 *Let* f *be a meromorphic function and* g *be an entire function such that* $0 < \lambda_g < \rho_f$, $0 < \lambda_f \leq \rho_f < \infty$, $\overline{\sigma}_f > 0$ *and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a\,;f) = 4$. *Then*

$$\liminf_{r \to \infty} \frac{\log T(r,fog)}{T(r\,,M[\,f\,])} \leq \min \frac{1}{\{\Gamma_M - (\Gamma_M - \gamma_M)\ \Theta(\infty\,;g)\}} \left\{ \frac{\rho_f}{\overline{\sigma}_f}, \frac{\rho_g}{\overline{\sigma}_f} \right\}.$$

Theorem 12 *Let* f *be a meromorphic function and* g *be a transcendental entire function such that* $0 < \lambda_g < \rho_f$, $0 < \lambda_f \leq \rho_f < \infty$, $\overline{\sigma}_g > 0$ *and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a\,;g) \leq 4$. *Then*

$$\liminf_{r \to \infty} \frac{\log T(r,fog)}{T(r\,,M[\,g\,])} \leq \min \frac{1}{\{\Gamma_M - (\Gamma_M - \gamma_M)\ \Theta(\infty\,;g)\}} \left\{ \frac{\rho_f}{\overline{\sigma}_g}, \frac{\rho_g}{\overline{\sigma}_g} \right\}.$$

Using the notion of weak type , we may state the following theorem without proof :

Theorem 13 *Let* f *be a meromorphic function and* g *be an entire function such that* (*i*) $0 < \lambda_f \leq \rho_f < \infty$, (*ii*) $\lambda_f = \lambda_g$, (*iii*) $\overline{\tau}_g < \infty$, (*iv*) $\tau_f > 0$ *and* $\Theta(\infty\,;f) = \sum_{a \neq \infty} \delta_p(a\,;f) = 1$ *or* $\delta(\infty\,;f) = \sum_{a \neq \infty} \delta(a\,;f) = 1$. *Then*

$$\liminf_{r \to \infty} \frac{\log T(r,fog)}{T(r\,,P_0[\,f\,])} \leq \frac{1}{\gamma_{P_0[\,f\,]}} \min \left\{ \rho_f \frac{\tau_g}{\tau_f}, \rho_f \frac{\overline{\tau}_g}{\overline{\tau}_f}, \lambda_f \frac{\overline{\tau}_g}{\tau_f} \right\}.$$

Remark 12 *if we take* $\sum_{a \neq \infty} \Theta(a\,;f) = 2$ *instead of* $\Theta(\infty\,;f) = \sum_{a \neq \infty} \delta_p(a\,;f) = 1$ *or* $\delta(\infty\,;f) = \sum_{a \neq \infty} \delta(a\,;f) = 1$ *in Theorem* 13 *and the other conditions remain the same then Theorem* 13 *remain valid with* $\Gamma_{P_0[\,f\,]}$ *instead of* $\gamma_{P_0[\,f\,]}$.

Theorem 14 *Let* f *be a transcendental meromorphic function and* g *be an entire function such that* (*i*) $0 < \lambda_f \leq \rho_f < \infty$, (*ii*) $\lambda_f = \lambda_g$, (*iii*) $\overline{\tau}_g < \infty$, (*iv*) $\tau_f > 0$ *and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a\,;f) \leq 4$. *Then*

$$\liminf_{r \to \infty} \frac{\log T(r,fog)}{T(r\,,M[\,f\,])} \leq \frac{1}{\{\Gamma_M - (\Gamma_M - \gamma_M)\ \Theta(\infty\,;f)\}} \min \left\{ \rho_f \frac{\tau_g}{\tau_f}, \rho_f \frac{\overline{\tau}_g}{\overline{\tau}_f}, \lambda_f \frac{\overline{\tau}_g}{\tau_f} \right\}.$$

The proof is omitted as it can be carried out in the line of Theorem 13 and in view of Lemma 12 .

In the line of Theorem 7 we may state the following theorem without proof .

Theorem 15 *Let* f *be a meromorphic function and* g *be an entire function such that* (i) $0 < \lambda_f < \infty$, (ii) $\overline{\tau}_g < \infty$, (iii) $\tau_f > 0$ *and* $\Theta(\infty\,;g) = \sum_{a \neq \infty} \delta_p(a\,;g) = 1$ *or* $\delta(\infty\,;g) = \sum_{a \neq \infty} \delta(a\,;g) = 1$. *Then*

$$\liminf_{r \to \infty} \frac{\log T(r, fog)}{T(r\,, P_0[\,g\,])} \leq \frac{\lambda_f}{\gamma_{P_0[\,g\,]}} \cdot \frac{\overline{\tau}_g}{\tau_g} .$$

Remark 13 *if we take* $\sum_{a \neq \infty} \Theta(a\,;g) = 2$ *instead of* $\Theta(\infty\,;g) = \sum_{a \neq \infty} \delta_p(a\,;g) = 1$ *or* $\delta(\infty\,;g) = \sum_{a \neq \infty} \delta(a\,;g) = 1$ *in Theorem* 15 *and the other conditions remain the same then Theorem* 15 *is still valid with* $\Gamma_{P_0[\,g\,]}$ *instead of* $\gamma_{P_0[\,g\,]}$.

Remark 14 *In addition to the conditions of Theorem* 15 *if* f *be a meromorphic function with* $0 < \lambda_f^{**} < \infty$ *then one can easily verify that*

$$\liminf_{r \to \infty} \frac{T(r, fog)}{T(r\,, P_0[\,g\,])} \leq \frac{\{1 + o(1)\}\,\lambda_f^{**}}{\gamma_{P_0[\,g\,]}} \cdot \frac{\overline{\tau}_g}{\tau_g} .$$

The following theorem can be carried out in the line of Theorem 15 and in view of Lemma 12 :

Theorem 16 *Let* f *be a meromorphic function and* g *be a transcendental entire function with* (i) $0 < \lambda_f < \infty$ (ii) $\overline{\tau}_g < \infty$, (iii) $\tau_f > 0$ *and* (iv) $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a\,;g) \leq 4$. *Then*

$$\liminf_{r \to \infty} \frac{\log T(r, fog)}{T(r\,, M[\,g\,])} \leq \frac{\lambda_f}{\{\Gamma_M - (\Gamma_M - \gamma_M)\;\Theta(\infty\,;g)\}} \cdot \frac{\overline{\tau}_g}{\tau_g} .$$

Remark 15 *In addition to the conditions of Theorem* 16 *if* f *be a meromorphic function with* $0 < \lambda_f^{**} < \infty$ *then one can easily verify that*

$$\liminf_{r \to \infty} \frac{T(r, fog)}{T(r\,, M[\,g\,])} \leq \frac{\{1 + o(1)\}\,\lambda_f^{**}}{\{\Gamma_M - (\Gamma_M - \gamma_M)\;\Theta(\infty\,;g)\}} \cdot \frac{\overline{\tau}_g}{\tau_g} .$$

Theorem 17 *Let* f *be a meromorphic function and* g *be an entire function such that* $0 < \lambda_g < \lambda_f$, $0 < \lambda_f \leq \rho_f < \infty$, $\tau_f > 0$ *and* $\Theta(\infty\,;f) = \sum_{a \neq \infty} \delta_p(a\,;f) = 1$ *or* $\delta(\infty\,;f) = \sum_{a \neq \infty} \delta(a\,;f) = 1$. *Then*

$$\liminf_{r\to\infty}\frac{\log T(r,fog)}{T(r\,,P_0[f\,])} \leq \frac{1}{\gamma_{P_0[f\,]}}\min\left\{\frac{\rho_f}{\tau_f}\,,\frac{\rho_g}{\tau_f}\right\}.$$

The proof of the Theorem is omitted because it can be carried out in the line of Theorem 9 and using the notion of weak type .

Remark 16 *if we take* $\sum_{a\neq\infty}\Theta(a\,;f)=2$ *instead of* $\Theta(\infty\,;f)=\sum_{a\neq\infty}\delta_p(a\,;f)=1$ *or* $\delta(\infty\,;f)=\sum_{a\neq\infty}\delta(a\,;f)=1$ *in Theorem* 17 *and the other conditions remain the same then Theorem* 17 *is also valid with* $\Gamma_{P_0[f\,]}$ *instead of* $\gamma_{P_0[f\,]}$.

Theorem 18 *Let* f *be a meromorphic function and* g *be an entire function such that* $0<\lambda_g<\lambda_f$, $0<\lambda_f\leq\rho_f<\infty$, $\tau_f>0$ *and* $\sum_{a\,\epsilon\,\mathbb{C}\,\cup\{\infty\}}\delta_1(a\,;f)=4$. *Then*

$$\liminf_{r\to\infty}\frac{\log T(r,fog)}{T(r\,,M[f\,])} \leq \frac{\min\left\{\frac{\rho_f}{\tau_f},\ \frac{\rho_g}{\tau_f}\right\}}{\{\Gamma_M-(\Gamma_M-\gamma_M)\ \Theta(\infty\,;f)\}}.$$

We omit the proof of Theorem 18 because it can be carried out in the line of Theorem 17 .

Theorem 19 *Let* f *be a meromorphic function and* g *be an entire function such that* $0<\lambda_g<\lambda_f$, $0<\lambda_f\leq\rho_f<\infty$, $\tau_g>0$ *and* $\Theta(\infty\,;g)=\sum_{a\neq\infty}\delta_p(a\,;g)=1$ *or* $\delta(\infty\,;g)=\sum_{a\neq\infty}\delta(a\,;g)=1$. *Then*

$$\liminf_{r\to\infty}\frac{\log T(r,fog)}{T(r\,,P_0[g])} \leq \frac{1}{\gamma_{P_0[g\,]}}\min\left\{\frac{\rho_f}{\tau_g}\,,\frac{\rho_g}{\tau_g}\right\}.$$

The proof of Theorem 19 is omitted because it can be carried out in the line of Theorem 17.

Remark 17 *if we take* $\sum_{a\neq\infty}\Theta(a\,;g)=2$ *instead of* $\Theta(\infty\,;g)=\sum_{a\neq\infty}\delta_p(a\,;g)=1$ *or* $\delta(\infty\,;g)=\sum_{a\neq\infty}\delta(a\,;g)=1$ *in Theorem* 19 *and the other conditions remain the same then Theorem* 19 *remain valid with* $\Gamma_{P_0[g\,]}$ *instead of* $\gamma_{P_0[g\,]}$.

Theorem 20 *Let* f *be a meromorphic function and* g *be an entire function with* $0<\lambda_g<\lambda_f$, $0<\lambda_f\leq\rho_f<\infty$, $\tau_g>0$ *and* $\sum_{a\,\epsilon\,\mathbb{C}\,\cup\{\infty\}}\delta_1(a\,;g)\leq 4$. *Then*

$$\liminf_{r\to\infty}\frac{\log T(r,fog)}{T(r\,,M[g\,])} \leq \frac{\min\left\{\frac{\rho_f}{\tau_g},\ \frac{\rho_g}{\tau_g}\right\}}{\{\Gamma_M-(\Gamma_M-\gamma_M)\ \Theta(\infty\,;g)\}}.$$

The proof is omitted .

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