Ultra-distributions associated with Fourier-Hankel type transformation

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Abstract: In this paper we study the Fourier-Hankel type transform to spaces of Ultra-distributions. For this purpose, spaces $F\,H_{\alpha,\beta,a_k,A,a_k',A'}$, $F\,H_{\alpha,\beta}^{b_q,B,b_q',B'}$, $F\,H_{\alpha,\beta,a_k,A,a_k',A'}^{b_q,B,b_q',B'}$ are constructed on which Fourier-Hankel type transform $\left(Fh_{\alpha,\beta}\right)$ is defined. It is proved that the so defined F-H type transform $Fh_{\alpha,\beta}$ is a continuous linear mapping from the space $F\,H_{\alpha,\beta,a_k,A,a_k',A'}^{b_q,B,b_q',B'}$ into the space $F\,H_{\alpha,\beta,b_k,B,a_k',b_k',A_1'}^{a_q,A,a_q^2,B_1'}$. Further generalized F-H transform is defined and its inversion formula is given. An operational transform formula is also established. In the end, a differential equation of the form $P\left(D_x,\Delta_{\alpha,\beta}\right)u = g$ has been solved by using the so defined F-H transform.

Keywords: Fourier-Hankel type transformation, Ultra-distribution.

1. Introduction:

If the test function spaces are some classes of non-quasi-analytic functions with some natural topology, then the dual spaces have the properties analogous to those of distributions (see Pathak [4]). The elements of these spaces are the ultra-distributions. Pathak [4] has given a comprehensive account of extensions of Fourier and Hankel transformations of Ultra-distributions (of Roumieu type). Following the idea of Roumieu [5] and Komatsu [2], we introduce the space of Ultra-differentiable functions on which the combined Fourier-Hankel transformation acts as a continuous linear mapping, so that the generalized $F-H$ transformation on the corresponding dual spaces also acts as a continuous linear mapping.

2. Notations and terminology:

In this paper we follow the notations and terminology of Pathak [4] and Zemanian [6].

Fourier-Hankel type transform: We define an integral transform for which the Kernel is the product of the kernels of Fourier and the Hankel transformations as below:

Let $\phi\,(x,y)$ be a suitably restricted function on $-\infty < x < \infty,\ 0 < y < \infty$ then its Fourier-Hankel type transform is given by:

$$F\,h_{\alpha,\beta}\,\phi = \Phi\,(\lambda,t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{i\,\lambda x}\,(yt)^{\alpha+\beta}\,J_{\alpha-\beta}(yt)\,\phi(x,y)\,dxdy\,, \tag{2.1}$$

where $J_{\alpha-\beta}\,(yt)$ is the Bessel type function of first kind of order $\alpha-\beta$, and $\alpha-\beta$ is real with $(\alpha-\beta) \geq -\frac{1}{2}$.

3. Test function spaces and their duals:

Let $\{a_k\}$ and $\{b_k\}$ be two arbitrary sequences of positive real numbers. We shall impose some of the following constraints on these sequences so that the resulting space of test functions may be non-quasi analytic and closed under certain algebraic, differential and integral operations.

$$b_q^2 \leq b_{q-1}\, b_{q+1} \;\; for \;\; all \;\; q \in \mathbb{N}_0 \,. \tag{3.1}$$

An immediate consequence of this inequality is

$$b_p\, b_q \leq b_0\, b_{p+q}\,, \quad p,q \in \mathbb{N}_0 \;; \tag{3.2}$$

and

$$\textstyle\sum_{q=0}^{\infty} b_{q-1}\, b_q < \infty \quad . \tag{3.3}$$

Further there are constants $R,\ R_1 > 0$ and $H, H_1 > 1$ such that

$$b_p \leq R\, H^p \, \min_{0\leq q\leq p} b_p\, b_{p-q} \quad , \quad p \in \mathbb{N}_0 \tag{3.4}$$

and

$$a_p \leq R_1\, H_1^p \, \min_{0\leq q\leq p} a_q\, a_{p-q}\,, \; p \in \mathbb{N}_0 \tag{3.5}$$

Now we construct certain test function spaces on which $F-H$ transformation can be studied systematically.

The test function spaces:

$$F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}} \;,\; F\,H_{\alpha,\beta}^{b_q,B,b_q^{'},B^{'}} \;,\; F\;H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}} \;.$$

Let ϕ be an infinitely differentiable function defined on the set

$$I = (-\infty,\infty) \times (0,\infty)\,.$$

$\phi \in F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$ if and only if

$$\left| x^k\, D_x^q\, y^{k^{'}} \left(y^{-1}\, D_y\right)^{q^{'}} \, y^{2\beta-1}\, \phi\,(x,y) \right| \leq C^{\alpha-\beta}\, (A+\delta)^k\, (B+\rho)^q\, (A^{'}+\delta)^k$$
$$\times\, (B^{'}+\rho^{'})^q\, a_k\, b_q\, a_k^{'}\, b_q^{'}$$

for all $k, k^{'},\ q, q^{'} \in \mathbb{N}_0$, where $\delta, \delta^{'}$ and $\rho, \rho^{'} > 0$ are arbitrary small numbers and $C^{\alpha-\beta}$, $A, B, A^{'}, B^{'}$ are certain positive constants depending on ϕ and $\{a_k\}$, $\{b_k\}$ are arbitrary sequences of positive numbers satisfying the conditions (3.1) to (3.5) for ascertaining that the resultant space of test functions is non-quasi analytic and closed under certain algebraic differential and integral operations.

In this space, we introduce the norm as follows:

$$\|\phi\|^{\alpha,\beta}_{\delta,\delta',\rho,\rho'} = Sup \frac{\left|x^k D_x^1 y^{k'} \left(y^{-1}D_y\right)^{q'} y^{2\beta-1} \phi(x,y)\right|}{(A+\delta)^k (B+\rho)^q (A'+\delta')^{k'} (B'+\rho')^{q'} a_k b_q a'_k b'_q} \tag{3.6}$$

where Sup is over all

$$(x,y) \in (-\infty,\infty) \times (0,\infty),\ k,\ k', q,\ q' \in \mathbb{N}_0 \ .$$

Here

$$q, q', \delta, \delta', \rho, \rho' = 1, \frac{1}{2}, \dots \dots$$

Here we note that

$$\|\phi\|^{\alpha,\beta}_{1/n} \le \|\phi\|^{\alpha,\beta}_{1/(n+1)},\ n = 1,2, \dots \dots$$

Similarly the other spaces $F\,H_{\alpha,\beta,a_k,A,a'_k,A'}$, $H^{b_q,B,b'_q,B'}_{\alpha,\beta}$ can be defined and corresponding norms on each of them, as follows :

$$\phi \in F\,H_{\alpha,\beta,a_k,A,a'_k,A'} \text{ if and only if}$$

$$\left|x^k D_x^q y^{k'} \left(y^{-1} D_y\right)^{q'} y^{2\beta-1} \phi(x,y)\right| \le C^{\alpha-\beta} (A+\delta)^k (A'+\delta')^{k'} a_k a'_k$$

and

$$\phi \in F\,H^{b_q,B,b'_q,B'}_{\alpha,\beta} \text{ if and only if}$$

$$\left|x^k D_x^q y^{k'} \left(y^{-1}D_y\right)^{q'} y^{2\beta-1} \phi(x,y)\right| \le C^{\alpha-\beta}(B+\delta)^q (B'+\rho')^{q'} b_q b'_q , and$$

$$\phi \in FH^{b_q,\beta,b'_q,\beta'}_{\alpha,\beta} \text{ if and only if}$$

$$\left|x^k D_x^q y^{k'} \left(y^{-1}D_y\right)^{q'} y^{2\beta-1} \phi(x,y)\right| \le C^{\alpha-\beta}(B+\delta)^q (B'+\rho')^{q'} b_q b'_q , \ .$$

For $a_k = k^{kr}$, $a'_k = k'^{k'r}$ $and\ b_q = q^{qs}, b'_q = q'^{q's}$ $r, r', s, s' \ge 0$,

it can be seen that the spaces $F\,H_{\alpha,\beta,a_k,A,a'_k,A'}$, $F\,H^{b_q,B,b'_q,\ B'}_{\alpha,\beta}$, $FH^{b_q,B,b'_q,B'}_{\alpha,\beta,a_k,A,a'_k,A'}$ reduce to $F\,H_{\alpha,\beta,r,r',A,A'}$, $FH^{s,s',B,B'}_{\alpha,\beta}$, $F\,H^{\beta,\beta' B,B'}_{\alpha,\beta,r,r' A,A'}$ respectively, similar to the those studies by Lee [3].

If b_q , b'_q satisfy the condition (3.1), then the space $D\left\{b_q ,\ b'_q(-\infty,\infty)\times(0,\infty)\right\}$ is a subspace of $F\,H^{b_q,B,b'_q,B'}_{\alpha,\beta,a_k,A,a'_k,A'}$ and the convergence in $D\left\{b_q, b'_q, (-\infty,\infty)\times(0,\infty)\right\}$ implies convergence in $F\,H^{b_q,b_q,B,b'_q,B'}_{\alpha,\beta,a_k,A,a'_k,A'}$.

Following Gel'fand and Shilov [1, pp.179-181], we prove the following theorem:

Theorem 3.1: Let $a_k, a_k^{'}$ satisfy (3.5). Then $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$ is a complete countably normed perfect space. The dual is also complete.

Let $\phi\,(x,y)$ be an infinitely differentiable function defined on $-\infty < x < \infty$, $0 < y < \infty$.

$\phi\,(x,y) \in F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$ if and only if (3.6) holds.

With the system of norms (3.6), we assert that the space $F\,H_{\alpha,\beta,a_k,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$ becomes a complete countably normed space. All that we need here is to establish that for every Cauchy sequence $\{\phi_\nu\,(x,y)\}$ in $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$, $\{D^k\,\phi_\nu\,(x,y)\}$ converges uniformly on every compact subset of $R \times I$ to smooth function $D^k\,\phi\,(x,y)$, for each $k = 1,2,\ldots\ldots$, where $\phi\,(x,y) \in F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$.

Now, the convergence of $\{\phi_\nu(x,y)\}$ can be defined as follows:

Definition 3.1: A sequence of an infinitely differentiable function $\{\phi_\nu\,(x,y)\}$ is said to be correctly convergent to the function $\phi\,(x,y)$ if for any $q, q^{'}$, the function $x^k\,D_x^q\,y^{k^{'}}\left(y^{-1}\,D_y\right)^{q^{'}}\,y^{2\beta-1}\,\phi_\nu\,(x,y)$ converges uniformly to $x^k\,D_x^q\,y^{k^{'}}\,\left(y^{-1}\,D_y\right)^{q^{'}}\,y^{2\beta-1}\,\phi_\nu\,(x,y)$ in any bounded interval.

The proof of the theorem which runs parallel to that of one given by Pathak $[4, pp. 286 - 289]$ is broken into several steps:

I. If the sequence $\{\phi_\nu(x,y)\}$ converges correctly to some function $\phi\,(x,y)$ and for some $\delta, \rho, \delta^{'}, \rho^{'}$,

$$\|\phi_\nu\|_{\delta^{'},\rho^{'}}^{\delta,\rho} \leq C^{\alpha-\beta} \quad , \qquad C^{\alpha-\beta} > 0 \ ,$$

then the norm $\|\quad\|_{\delta^{'},\rho^{'}}^{\delta,\rho}$ exists even for some function $\phi\,(x,y)$ and

$$\|\phi_\nu\|_{\delta^{'},\rho^{'}}^{\delta,\rho} \leq C^{\alpha-\beta}.$$

Now for $-a < x < a, \quad 0 \leq y < b$,

$$\sup_{x,y} \frac{\left|x^k D_x^q y^{k^{'}} \left(y^{-1}\,D_y\right)^{q^{'}}\, y^{2\beta-1}\,\phi(x,y)\right|}{(A+\delta)^k\,(B+\rho)^q\,(A^{'}+\delta^{'})^{k^{'}}(B^{'}+\rho^{'})^{q^{'}}\,a_k\,b_q\,a_k^{'}\,b_q^{'}} \quad , \qquad k, k^{'} \leq q, q^{'} \leq p$$

$$= \lim_{\nu \longrightarrow \infty} \sup_{x,y} \frac{\left|x^k D_x^q\,y^{k^{'}}\,\left(y^{-1} D_y\right)^{q^{'}}\,y^{2\beta-1}\,\phi(x,y)\right|}{(A+\delta)^k\,(B+\rho)^q (A^{'}+\delta')^{k^{'}}(B^{'}+\rho^{'})^{q^{'}}\,a_k b_k a_k^{'} b_k^{'}}$$

$\leq \ \|\phi_\nu\|_{\delta' \rho'}^{\delta,\rho}$

$\leq \ C^{\alpha-\beta}$.

Now, we take the limit $a \longrightarrow \infty,\ b \longrightarrow \infty,\ p \longrightarrow \infty$, and obtain

$$\underset{x,y}{Sup} \frac{\left|x^k D_x^q \ y^{k'} \left(y^{-1} D_y\right)^{q'} \ y^{2\beta-1} \ \phi\ (x,y)\right|}{(A+\delta)^k \ (B+\rho)^q \ (A'+\delta')^{k'} (B'+\rho')^{q'} a_k \ b_q \ a_k' b_q'} \leq \ C^{\alpha-\beta} \quad ,$$

$$k, k' \ \leq q, q'$$

II. If the sequence $\{\phi_\nu(x,y)\}$ converges to zero at each point and is fundamental in the norm $\|.\|_{\delta',\rho'}^{\delta,\rho}$, then $\|\phi_\nu\|_{\delta',\rho'}^{\delta,\rho} \ \longrightarrow 0$.

As the sequence $\{\phi_\nu\}$ is fundamental, it converges correctly to zero and hence the sequence $\{\phi_\nu - \phi_\mu\}$ converges correctly to ϕ_ν as $\mu \ \longrightarrow \ \infty$.

Thus for given $\epsilon > 0$ there exists a sufficiently large ν such that

$$\|\phi_\nu\|_{\delta',\rho'}^{\delta,\rho} \ \leq \ \underset{\mu \geq \nu}{Sup} \ \|\phi_\nu - \phi_\mu\|_{\delta',\rho'}^{\delta,\rho} \ < \ E \ .$$

III. The space $F\ H_{\alpha,\beta,a_k,A,a_k',A'}^{b_q,B,b_q',B'}$ be a fundamental sequence in each of the norms $\|.\|_{\delta',\rho'}^{\delta,\rho}$. Then according I each of the norms $\|.\|_{\delta',\rho'}^{\delta,\rho}$ exists for limit function $\phi\ (x,y)$; hence

$$\phi\ (x,y) \ \in F\ H_{\alpha,\beta,a_k,A,a_k',A'}^{b_q,B,b_q',B'} \ .$$

Also, according to II, the difference $\{\phi - \phi_\nu\}$ converges correctly to zero and is bounded in each of the norms.

Hence, we have

$$\|\phi - \phi_\nu\|_{\delta',\rho'}^{\delta,\rho} \longrightarrow 0 \quad for\ any\ q,\ q' \ .$$

Thus, the space $F\ H_{\alpha,\beta,a_k,A,a_k',A'}^{b_q,B,b_q',B'}$ is complete.

IV. The norms $\| \quad \|_{\delta',\rho'}^{\delta,\rho}$ are pairwise consistent.

Let $\eta > 0, \delta, \delta'$ and $\rho, \rho' > 0$ be given and choose arbitrarily $\delta' < \delta\ ,\ \rho'' < \rho',\ \delta' < \delta, \rho' < \rho$.

Let $\{\phi\ (x,y)\} \ \in \ F\ H_{\alpha,\beta,a_k,A,a_k',A'}^{b_q,B,b_q',B'}$ be fundamental in $\|.\|_{\delta',\rho''}^{\delta,\rho}$. Since $\phi_\nu(x,y)$ is bounded with respect to $\| \quad \|_{\delta',\rho''}^{\delta,\rho}$, for any k, k', q, q' and x, y , we have

$$\left|x^k\ \mathcal{D}_x^q\ y^{k'}\ \left(y^{-1} D_y\right)^{q'}\ y^{2\beta-1}\ \phi(x,y)\right| \ \leq \ (A+\delta)^k\ (B+\rho)^q\ (A'+\delta')^{k'}\ (B'+\rho')^{q'}$$

$$\times\ a_k\ b_q\ a_k^{'}\ b_q^{'}\ .$$

For sufficiently large $k\ >\ k_0,\ k^{'}\ > k_0^{'}$, the inequality

$(A+\delta^{'})^k\ (A^{'}+\delta')^{k^{'}}\ \leq\ \left(\eta' C_1^{\alpha-\beta}\right)(A+\delta)^k\ (A^{'}+\delta^{'})^{k^{'}}$ holds.

Consequently, for any $q, q^{'}, x, y$ and $k\ \geq\ k_0,\ k^{'}\geq k_0^{'}$,

$$\left|x^k\ D_x^q\ y^{k^{'}}\ \left(y^{-1}\ D_y\right)^{q^{'}}\ y^{2\beta-1}\ \phi_\nu\ (x,y)\right|\ \leq\ \eta\ (A+\delta)^k\ (B+\rho)^q\ (A^{'}+\delta^{'})^{k^{'}}\ (B^{'}+\rho^{'})^{q^{'}}$$

$$\times\ a_k\ b_q\ a_k^{'}\ b_q^{'}\ . \tag{3.2}$$

Next, using boundedness of $\phi_\nu(x,y)$ with respect to $\|.\|_{\delta^{'},\rho^{''}}^{\delta,\rho}$, we arrive at (3.7), for any $k, k^{'}, x, y\ \ and\ q\ \geq\ q_0,\ q^{'}\ \geq\ q_0^{'}$.

We now examine the remaining case when $k\ <\ k_0$ and $k^{'}\ <\ k_0^{'}$, $q\ <\ q_0$, $q^{'}\ <\ q_0^{'}$. For $k\ <\ k_0$, $k^{'}\ <\ k_0^{'}$, $|x|\ > 1,\ |y|\ > 1,$ we have for any $q, q^{'}$ and x, y by virtue of (3.7),

$$\left|x^k\ D_x^q\ y^{k^{'}}\ \left(y^{-1} D_y\right)^{q^{'}}\ y^{2\beta-1}\ \phi_\nu\ (x,y)\right|$$

$$=\ \frac{|x|^{k_0}\ |y|^{k_0^{'}}}{|x|^{k_0-k}|y|^{k_0^{'}-k^{'}}}\ \left|D_x^q\ \left(y^{-1}D_y\right)^{q^{'}} y^{2\beta-1}\ \phi_\nu\ (x,y)\right|$$

$$\leq\ \frac{1}{|x|\ |y|}\ \eta\ (A+\delta)^{k_0}\ (B+\rho)^q\ (A^{'}+\delta^{'})^{k_0^{'}}\ (B^{'}+\rho^{'})^{q^1}\ a_{k_0}\ b_q\ a_{k_0}^{'}\ b_q^{'}\ .$$

For sufficiently large $|x|$, say $|x| >\ x_0$, and $|y|$, say $|y| >\ y_0$, we obtain

$$\frac{(A+\delta)^{k_0}}{|x|}\ \frac{(A^{'}+\delta^{'})^{k_0^{'}}}{|y|}\ a_{k_0}\ b_q\ a_{k_0}^{'} b_q^{'}\ \leq\ (A+\delta)^k\ (A^{'}+\delta^{'})^{k^{'}} a_k\ b_q\ a_k^{'}\ b_k^{'}$$

$$(k^{'}\ = 0,1,2, \ldots\ldots, k_0^{'}, q^{'}\ = 1,2, \ldots\ldots q_0^{'} - 1)$$

and therefore for $q\ <\ q_0,\ q^{'}\ < q_0^{'}$, $k\ < k_0, k^{'}\ <\ k_0^{'}$, the inequality (3.7) is satisfied.

Finally, if $k\ <\ k_0,\ k^{'}\ < k_0^{'}$ then for fixed $\delta, \delta^{'}, \rho, p'$ constants $(A+\delta)^k\ (B+\rho)^q\ (A^{'}+\delta')^{k^{'}}\ (B^{'}+\rho^{'})^q\ a_k\ b_q\ a_k^{'}\ b_q^{'}$ are bounded by some number C_2 .

Since the sequence $\left\{\left|D_x^q\ \left(y^{-1}\ D_y\right)^{q^{'}}\ y^{2\beta-1}\ \phi_\nu\ (x,y)\right| \longrightarrow 0\right.$ for $-\ x_0 \leq x \leq x_0$, $0 \leq y \leq y_0\ as\ \nu \longrightarrow\ \infty$ for given $\eta > 0$ there exists ν_0 sufficiently large such that for $\nu > \nu_0$, the inequality (3.7) holds. Then, for $\nu > \nu_0$, the inequality (3.7) is satisfied for all $x, y, k, k^{'}, q, q^{'}$.

Consequently, for $\nu > \nu_0$, $\|\phi_\nu\|_{\delta^{'},\rho^{'}}^{\delta,\rho}\ \leq\ \eta$, from which it also follows that the sequence $\{\phi_\nu\}$ tends to zero in the topology of the space $F\ H_{\alpha,\beta,a_k,a_k^{'},A^{'}}^{b_q,\beta,b_q^{'},B^{'}}\ as\ \nu\ \longrightarrow\ \infty$.

V. If the sequence $\{\phi_\nu(x,y)\}$ is bounded in each of the norms $\|\phi_\nu\|_{\delta',\rho'}^{\delta,\rho}$ and converges correctly to zero, it tends to zero in the topology of the space $FH_{\alpha,\beta,a_k,A,a_k',A'}^{bq,B,b_q',B'}$.

Let $\delta,\delta',\rho,\rho'$ and an arbitrary $\eta > 0$ be given. Choose $\delta' < \delta, \rho' < \rho, \delta'' < \delta, \rho'' < \rho'$.
The numbers $\|\phi_\nu\|_{\delta',\rho'}^{\alpha,\beta,\delta,\rho}$ are bounded by the constant $C_{\delta',\rho''}^{\alpha,\beta,\delta,\rho}$.
For sufficiently large q, q', k, k' say $q_0 \geq q_0'$, $k_0 \geq k_0'$ respectively, in the inequality.

$$\frac{(A+\delta)^k\,(B+\rho')^q\,(A'+\delta')^{k'}\,(B'+\rho'')^{q'}}{(A+\delta)^k\,(B+\rho)^q\,(A'+\delta')^{k'}\,(B'+\rho')^{q'}} < \frac{\eta}{C_{\delta',\rho'}^{\alpha,\beta,\delta,\rho}}\ \ holds\ .$$

Hence, for $k \leq k_0, k' \leq k_0'$, $q \leq q_0$, $q' \geq q_0'$, we have

$$\left|x^k\,D_x^q\,y^{k'}\,\left(y^{-1}D_y\right)^{q'}\,y^{2\beta-1}\,\phi_\nu(x,y)\right| \leq \eta\,(A+\delta)^k\,(B+\rho)^q\,(A'+\delta')^{k'}$$

$$\times\ (B'+\rho')^{q'}\,a_k\,b_q\,a_k'\,b_q'\ .$$

For $k \leq k_0,\ k' \leq k_0',\ q \leq q_0,\ q' \geq q_0'$ respectively and

$$|x|\,|y| > H_1^{k_0+k_0'}\left(C_{\delta',\rho''}^{\alpha,\beta,\delta,\rho}/\eta\right),\ \text{where}$$

$$C_{\delta',\rho''}^{\alpha,\beta,\delta,\rho} = a_1\,R_1\,H_1\ C_{\delta',\rho''}^{\alpha,\beta,\delta,\rho}\,(A+\delta)\,(A'+\delta')\ .$$

We have

$$\left|x^k\,D_x^q\,y^{k'}\,\left(y^{-1}\,D_y\right)^{q'}\,y^{2\beta-1}\,\phi(x,y)\right|$$

$$= \frac{1}{|x|\,|y|}\ \frac{\left|x^{k+1}\,D_x^q\,y^{k+1}\left(y^{-1}\,D_y\right)^{q'}\,y^{2\beta-1}\,\phi_\nu(x,y)\right|}{(A+\delta)^{k+1}\,(B+\rho)^q\,(A'+\delta')^{k+1}\,(B'+\rho')^{q'}\,a_{k+1}\,b_q}$$

$$\times\ (A+\delta)^{k+1}\,(B+\rho)^q\,(A'+\delta')^{k'+1}\,(B'+\rho')^{q'}\,a_{k+1}\,b_q$$

$$\leq\ a_1\,R_1\,H_1^{k+k'+2}\,\|\phi\|_{\delta,\rho'}^{\alpha,\beta,\delta,\rho}\,(A+\delta)^k\,(B+\rho)^q\,(A'+\delta')^{k'}\,(B'+\rho')^{q'}$$

$$\times\ a_k\,b_q\,a_k'\,b_q'$$

$$\leq\ \eta\,(A+\delta)^k\,(B+\rho)^q\,(A'+\delta')^{k'}\,(B'+\rho')^{q'}\ a_k\,b_q\,a_k'\,b_q'\ .$$

Finally, if $k < k_0,\ k' < k_0',\ q < q_0,\ q' < q_0'$ and

$$|x|\,|y| \leq H_q^{k_0+k_0'}\left(C_{\delta',\rho''}^{\alpha,\beta,\delta,\rho}/\eta\right)$$

then by virtue of uniform convergence of the sequence

$$\left\{D_x^q\,\left(y^{-1}D_y\right)^{q'}\,y^{2\beta-1}\,\phi_\nu\,(x,y)\right\},\ \text{the inequality}$$

$$\left| x^k\, D_x^q\, y^{k'} \left(y^1 D_y\right)^{q'} y^{2\beta-1}\, \phi_\nu\,(x,y) \right|$$

$$\leq\ \eta\,(A+\delta)^k\,(B+\rho)^q (A'+\delta')^{k'} (B'+\rho')^{q'} a_k\, b_q\, a_k'\, b_q'$$

will also hold for sufficiently large $\nu \geq \nu_0$.

Therefore, for $\nu \geq \nu_0$, the inequality (3.3) holds for all x, y, k, k', q, q'

For $\nu \geq \nu_0$

$$\|\phi_\nu\|_{\delta',\rho'}^{\delta,\rho} = \sup_{\substack{x,y \\ q,q'}} \frac{\left| x^k\, D_x^q\, y^{k'} \left(y^{-1} D_y\right)^{q'} y^{2\beta 1}\, \phi(x,y) \right|}{(A+\delta)^k\,(B+\rho)^q\,(A'+\delta')^{k'} (B'+\rho')^{q'} a_k\, b_q\, a_k'\, b_q'} < \eta\ ,$$

from which it follows that $\phi_\nu(x,y) \longrightarrow 0$ in the topology of $F\,H_{\alpha,\beta,a_k,A,a_k',A'}^{b_q,\beta,b_q',B'}$.

VI. If the sequence $\{\phi_\nu\,(x,y)\}$ is bounded in each of the norms $\|.\|_{\delta',\rho'}^{\delta,\rho}$ and converges correctly to some function $\phi\,(x,y)$ then $\phi_\nu(x,y) \in F\,H_{\alpha,\beta,a_k,A,a_k',A'}^{b_q,\beta,b_q',B'}$ and $Q\,(x,y)$ is the limit of the sequence $\{\phi_\nu(x,y)\}$ in the topology of the space $F\,H_{\alpha,\beta,a_k,A,a_k',A'}^{b_q,\beta,b_q',B'}$.

Now, $\phi_\nu\,(x,y) \in F\,H_{\alpha,\beta,a_k,A,a_k',A'}^{b_q,\beta,b_q',B'}$ by virtue of I.

The difference $\{\phi(x,y) - \phi_\nu(x,y)\}$ is bounded in all the norms and converges to zero; according to II the difference converges to zero in the topology of the space $F\,H_{\alpha,\beta,a_k,A,a_k',A'}^{b_q,\beta,b_q',B'}$

Thus proof is completed.

Similarly the other spaces can also be shown to be the complete countable normed spaces. Moreover, by invoking the theorem due to Zemanian [6, pp.21-23], we infer that the corresponding dual spaces are also complete.

Now, we define a countable union space as follows:

Let $A_1 < A_2$ and $B_1 < B_2$; then the space $F\,H_{\alpha,\beta,a_k,A_1,a_k',A_1'}^{b_q,B_1,b_q',B_1'}$ is a subspace of $F\,H_{\alpha,\beta,a_k,A_2,a_k',A_2'}^{b_q,B_2,b_q',B_2}$.

Further, the convergence of a sequence $\{\phi_\nu\,(x,y)\}$ in $F\,H_{\alpha,\beta,a_k,A_1,a_k',A_1'}^{b_q,B_1,b_q',B_1'}$ implies the convergence in $F\,H_{\alpha,\beta,a_k,A_2,a_k',A_2'}^{b_q,B_2,b_q',B_2'}$.

Hence we may construct the union of countably normed spaces $F\,H_{\alpha,\beta,a_k,A,a_k',A'}^{b_q,B_1,b_q',B'}$ for all indices $A, B = 1, 2, \dots \dots$

This union coincides with the space $F\,H^{b_q,B_2,b_q',B_2'}_{\alpha,\beta,a_k,a_k',A_2'}$.

$$F\,H^{b_q,B,b_q',B'}_{\alpha,\beta,a_k,a_k'} = \bigcup_{A,B=1}^{\infty} F\,H^{b_q,B,b_q',B'}_{\alpha,\beta,a_k,A,a_k',A'} .$$

Similarly, we define

$$F\,H_{\alpha,\beta,a_k,a_k'} = \bigcup_{A=1}^{\infty} F\,H_{\alpha,\beta,a_k,a_k',A'} \quad and$$

$$H^{b_q,B,b_q'}_{\alpha,\beta} = \bigcup_{B=1}^{\infty} H^{b_q,B,b_q',B'}_{\alpha,\beta}$$

The elements of the spaces $F\,H_{\alpha,\beta,a_k,a_k'}$, $F\,H_{\alpha,\beta,a_k,A,a_k',A'}$, $H^{b_q,B,b_q',B'}_{\alpha,\beta}$,$H^{b_q,B,b_q'}_{\alpha,\beta}$, $F\,H^{b_q,B,b_q',B'}_{\alpha,\beta,a_k,a_k'}$ are called ultra-differentiable functions and those of corresponding dual spaces are called ultra-distributions.

4. Differential and Integral Operators

Following Zemanian [6], we define the following operators.

$$N_{\alpha,\beta}\,\phi\,(x,y) = y^{2\alpha}\,D\,y^{2\beta-1}\,\phi(x,y)$$

$$M_{\alpha,\beta}\,\phi(x,y) = y^{2\beta-1}\,D\,y^{2\alpha}\,\phi(x,y)$$

$$N^{-1}_{\alpha,\beta}\,\phi(x,y) = y^{2\alpha}\int_{\infty}^{y} t^{2\beta-1}\,\phi(x,t)\,dt .$$

From which follows

$$\Delta_{\alpha,\beta} = M_{\alpha,\beta}\,N_{\alpha,\beta} = x^{2\beta-1}\,D\,x^{4\alpha}\,D\,x^{2\beta-1}$$

$$= (2\beta-1)\,(4\alpha+2\beta-2)\,x^{4(\alpha+\beta-1)} + 2(2\alpha+2\beta-1)\,x^{4\alpha+4\beta-3}D_x + x^{2(2\alpha+2\beta-1)}D_x^2 .$$

Note that for $\alpha = \frac{1}{4}+\frac{\mu}{2}$, $\beta = \frac{1}{4}-\frac{\mu}{2}$, $\Delta_{\alpha,\beta}$ reduces to

$$S_\mu = D^2 - (4\mu^2-1)/4x^2.$$

Now, we study these operators on the above spaces.

Theorem 4.1: The operation $\phi \longrightarrow N_{\alpha,\beta}\,\phi$ is a continuous linear mapping $F\,H_{\alpha,\beta,a_k,A,a_k',A'}$ into $F\,H_{\alpha,\beta,1,a_k,A,a_k',A'}$. If b_q' satisfies (3.1) then the operation $\phi \longrightarrow N_{\alpha,\beta}\,\phi$ is a continuous linear mapping from $F^{b_q,B,b_q',B'}_{\alpha,\beta,a_k,A,a_k',A'}$ *into* $F\,H^{b_q,B,b_q',B',H'}_{\alpha,\beta,1,a_k,A,a_k',A'}$.

Proof: For $\phi \in F\,H_{\alpha,\beta,a_k,A,a_k',A'}$, we have

$$\left|x^k D_x^q y^{k'} \left(y^{-1}D_y\right)^{q'} y^{2\beta-1} \left(N_{\alpha,\beta}\ \phi\,(x,y)\right)\right| = \left|x^k\, D_x^q\, y^{k'} \left(y^{-1}D_y\right)^{q'+1} y^{2\beta-1}\, \phi(x,y)\right|$$
$$\leq C\, C_q^{\alpha,\beta}\, (A+\delta)^k\, (A'+\delta')^{k'}\, a_k\, a_k'\ .$$

The proof of the remaining part is similar. Thus proof is completed.

Theorem 4.2: (a) Let a_k' satisfy (3.5) and b_q' satisfy (3.1) then the operation $\phi \longrightarrow M_{\alpha,\beta}\ \phi$ is a continuous linear mapping from $F\,H_{\alpha,\beta,1,a_k,a_k',A'}$ $\left(F\,H_{\alpha,\beta,1}^{b_q,B,b_q',B'}\right)$ into $F\,H_{\alpha,\beta,a_k,A,a_k',A',H_1'}$ $\left(F\,H_{\alpha,\beta}^{b_q,B,b_q',B',H'}\right)$.

(b) Let a_k' satisfy (3.5) and b_q' satisfy (3.1) then the operation $\phi \longrightarrow M_{\alpha,\beta}\ \phi$ is a continuous linear mapping from $F\,H_{\alpha,\beta,1,a_{k,A},a_k',A'}^{b_q,B,b_q',B'}$ into $F\,H_{\alpha,\beta,1,a_k,A,a_k',A'}^{b_q,B,b_q',B',H'}$.

Proof: For $\phi\ \in FH_{\alpha,\beta,1,a_k,A,a_k',A'}$, we have

$$\left|x^k\, D_x^q\, y^{k'}\, y^{2\beta-1}\left(y^{-1}D_y\right)^{q'} \left(M_{\alpha,\beta}\ \phi\,(x,y)\right)\right|$$
$$= \left|x^k D_x^q\, y^{k'}\left(y^{-1}D_y\right)^{q'+1} y^{2\beta-1}\phi(x,y)\right|$$
$$\leq C\, C_q^{\alpha,\beta}(A+\delta)^k\, (A'+\rho')^{k'}\, a_k\, a_k'\ .$$

Hence, the result follows. The proof of (b) is similar. Thus proof is completed.

Theorem 4.3: The operation $\phi \longrightarrow N_{\alpha,\beta}^{-1}\,\phi$ is a continuous linear mapping from $F\,H_{\alpha,\beta,1,a_k,A,a_k',A'}$ into $F\,H_{\alpha,\beta,a_k,A,a_k',A',H_1'}$. If b_q' satisfies (3.1) then the operation $\phi \longrightarrow N_{\alpha,\beta}^{-1}\ \phi$ is a continuous linear mapping $F\,H_{\alpha,\beta}^{b_q,B,b_q',B'}$ $\left(F\,H_{\alpha,\beta,1,a_k,A,a_A',A'}^{b_q,B,b_q',B'}\right)$ into $F\,H_{\alpha,\beta}^{b_q,B,b_q',B'}$ $\left(F\,H_{\alpha,\beta,1,a_k,A,a_A',A'}^{b_q,B,b_q',B'}\right)$.

Proof: We prove the last part of the theorem; the other two parts can be similarly proved. For $\phi\ \in F\,H_{\alpha,\beta,a_k,A,a_k',A'}^{b_q,B,b_q',B'}$, we have

$$\left|x^k D_x^q\, y^{k'}\left(y^{-1}D_y\right)^{q'} y^{2\beta-1}\ \left(N_{\alpha,\beta}^{-1}\ \phi\,(x,y)\right)\right|$$
$$= \left|x^k D_x^q y^{k'}\left(y^{-1}D_y\right)^{q'-1} y^{-(4\alpha+2\beta)}\ \phi(x,y)\right|$$
$$\leq C\, C_{q'}^{\alpha,\beta,1}\, (A+\delta)^k\, (B+\rho)^q\, (A'+\rho')^{k'}\, (B'+\rho')^{q'-1}\, a_k\, b_q\, a_k'\, b_{q-1}'$$
$$\leq\ C_{q'}^{\alpha,\beta,1}\, (A+\delta)^k\, (B+\rho)^q\, (A'+\rho')^{k'}\, (B'+\rho')^{q'}\, a_k\, b_q\, a_k'\, b_q'\ .$$

Theorems 4.1 and 4.2 imply that if $a_k^{'}$ satisfies (3.5) and $b_q^{'}$ satisfies (3.1) then the operation $\Delta_{\alpha,\beta}= M_{\alpha,\beta}\ .\ N_{\alpha,\beta}$ is a continuous linear mapping from $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$ into $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'},H^{'}}^{b_q,B,b_q^{'},B^{'},H^{'}}$.

Similar results hold for other two spaces also.

Operations in dual spaces:

In the dual spaces $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{'}$, $F\,H_{\alpha,\beta}^{b_q,B,b_q^{'},B^{'}}$, $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$, $N_{\alpha,\beta}$ is defined as the adjoint of $-M_{\alpha,\beta}$ and $M_{\alpha,\beta}$ is defined as the adjoint of $-N_{\alpha,\beta}$. More precisely, $N_{\alpha,\beta}$ is defined as a generalized differential operator on the above dual spaces by

$\langle N_{\alpha,\beta}\ f,\phi\rangle = \langle f,-M_{\alpha,\beta}\ \phi\rangle$, where ϕ belongs to $F\,H_{\alpha,\beta,1,a_k,A,a_k^{'},A}$ or $F\,H_{\alpha,\beta,1}^{b_q,B,b_q^{'},B^{'}}$ or $F\,H_{\alpha,\beta,1,a_k,A,a_k^{'},A^{'}}^{bq,B,b_q^{'},B^{'}}$ and f belongs to $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'},H_1^{'}}^{'}$ or $F\,H_{\alpha,\beta}^{b_q,B,b_q^{'},B^{'}}$ or $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'},H_1^{'}}^{b_q,B,b_q^{'},B^{'},H^{'}}$.

On the other hand, $M_{\alpha,\beta}$ is defined as a generalized differential operator on the dual spaces by

$\langle M_{\alpha,\beta}\ f,\phi\rangle = \langle f,-N_{\alpha,\beta}\ \phi\rangle$, where ϕ belongs to $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}$ or $F\,H_{\alpha,\beta}^{b_q,B,b_q^{'},B^{'}}$ or $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$, and f belongs to $F\,H_{\alpha,\beta,1,a_k,A,a_k^{'},A^{'},H_1^{'}}^{'}$ or $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'},H_1^{'}}^{b_q,B,b_q^{'},B^{'},H^{'}}$.

Now invoking theorem due to Zemanian [6, p.21-23] and Theorems 4.2 and 4.1 to the above definitions, we get

Theorem 4.4: (a) The operation $f\ \longrightarrow N_{\alpha,\beta}\ f$ is a continuous linear mapping $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'},H_1^{'}}$ into $F\,H_{\alpha,\beta,1,a_k,A,a_k^{'},A^{'}}$, if a_k satisfies (3.5), if $b_q^{'}$ satisfies (3.1).

$F\,H_{\alpha,\beta}^{b_q,B,b_q^{'},B^{'},H^{'}}$ $into$ $F\,H_{\alpha,\beta,1}^{b_q,B,b_q^{'},B^{'}}$ if bq satisfies (3.4), and of $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'},H^{'}}$ into $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$ if a_k satisfies (3.5) and b_q satisfies (3.4).

(b) The operation $f\ \longrightarrow M_{\alpha,\beta}\,f$ is a continuous linear mapping $F\,H_{\alpha,\beta,1,a_k,A,a_k^{'},A^{'}}$ onto $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}$, $F\,H_{\alpha,\beta,1}^{b_q,B,b_q^{'},B^{'},H^{'}}$ $into$ $F\,H_{\alpha,\beta}^{b_q,B,b_q^{'},B^{'}}$ if b_q satisfies (3.4), and if a_k satisfies (3.5), if $b_q^{'}$ satisfies (3.1), $F\,H_{\alpha,\beta}^{b_q,B,b_q^{'},B^{'},H}$ $into\ F\,H_{\alpha,\beta,1,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'},H^{'}}$ into $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$ if b_q satisfies (3.4).

(c) The operation $f \longrightarrow \Delta_{\alpha,\beta}\, f$ is a continuous linear mapping of $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'},H_1^{'}}$ onto $FH_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}$ if a_k satisfies (3.5), $F\,H_{\alpha,\beta}^{b_q,B,b_q^{'},B^{'},H^{'}}$ $into\ F\,H_{\alpha,\beta}^{b_q,B,b_q^{'},B^{'}}$ if b_q satisfies (3.4) , and if a_k satisfies (3.5), if $b_q^{'}$ satisfies (3.4) .

$$F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'},H_1}^{b_q,B,b_q^{'},B^{'},H^{'}}\, into\, F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}} \text{ if } b_q^{'} \text{ satisfies (3.4).}$$

5. Fourier-Hankel type transformation of test functions:

In this section we consider the mapping of the aforesaid ultra differentiable functions by $F\,h_{\alpha,\beta}$. It is easily seen that the Fourier-Hankel transform $F\,H_{\alpha,\beta}\ \phi$ exists for each test function ϕ in $F\,H_{\alpha,\beta,a_k,a_k^{'}}$, $F\,H_{\alpha,\beta\, a_k,A,a_k^{'},A^{'}}$, $F\,H_{\alpha,\beta}^{b_q,B,b_q^{'},B^{'}}$, $F\,H_{\alpha,\beta}^{b_q,B,b_q^{'}}$, $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$ when $(\alpha-\beta) \geq -\frac{1}{2}$.

Theorem 5.1: If a_k, $a_k^{'}$ satisfy the condition (3.5) then for $(\alpha-\beta) \geq -\frac{1}{2}$ the conventional Fourier-Hankel type transform $F\,h_{\alpha,\beta}$ defined by (2.1) is a continuous linear mapping from the space $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$ into the space $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{q_q,a_q^2,B_1^{'}}$ where $A_1^{'} = A^{'}\ B\ H_q^{'2}$ and $B_q^{'} = A^{'2}\ H_1^{'6}$.

Proof: Let K be a bounded set in $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{b_q,B,b_q^{'},B^{'}}$. Then every ϕ in K satisfies the inequality

$$\left|x^k\ D_x^q\ y^{k^{'}}\ \left(y^{-1}D_y\right)^{q^{'}}\ y^{2\beta-1}\ \phi\ (x,y)\right| \leq C^{\alpha,\beta}\ (A+\delta)^k\ (B+\rho)^q\ (A^{'}+\delta^{'})^{k^{'}}(B^{'}+\rho^{'})^{q^{'}}$$

$$\times\ a_k\ bq\ a_k^{'}\ b_q$$

for all $q, q^{'}\ \in\ \mathbb{N}_0$ and $k, k^{'} = 0,1,2\ldots\ldots$

Let $\Phi\ (\xi,t) = F\ h_{\alpha,\beta}\ \phi\ (x,y)$. For any pair of non-negative integers $k^{'}\ and\ q^{'}$, from Zemanian [6, p.139],

$$N_{\alpha,\beta,q^{'},k^{'}-1}\ \ldots\ldots N_{\alpha,\beta,q}\ \phi\ (x,y) = y^{q^{'}}\ N_{\alpha,\beta,q^{'}-1}\ \ldots..N_{\alpha,\beta}\ \phi\ (x,y) \text{ and using } q'\text{- times}$$

$$F\ h_{\alpha,\beta,1}\left(-y\ \phi\right) = N_{\alpha,\beta}\ F\ h_{\alpha,\beta}\ \phi\ \text{, we get}$$

$$N_{\alpha,\beta,q^{'},k^{'}-1}\ \ldots..N_{\alpha,\beta}\ \Phi\ =\ (-1)^{q^{'}}\ F\ h_{\alpha,\beta,q^{'}}\ \left(y^{q^{'}}\ \phi(x,y)\right).$$

Next apply $k^{'}$ −times, $F\ h_{\alpha,\beta}\left(N_{\alpha,\beta}\ \phi\right) = -(t)\ F\ h_{\alpha,\beta}\ \phi$, we get

$$N_{\alpha,\beta,q^{'},k^{'}-1}\ \ldots\ldots..N_{\alpha,\beta}\ \Phi = (-1)^{q^{'}}\ (-1/t)^{k^{'}}F\ h_{\alpha,\beta,q^{'},k^{'}-1}\ \ldots\ldots.N_{\alpha,\beta}\ \phi\ (x,y)).$$

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So that

$$(-t)^{k'} N_{\alpha,\beta,q',k'-1} \dots\dots N_{\alpha,\beta}\, \Phi(\xi,t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} (-y)^{q'} \left[N_{\alpha,\beta,q',k'-1} \dots\dots N_{\alpha,\beta}\,\phi\,(x,y)\right]$$

$$\times\; e^{i\xi x}\; y^{\alpha+\beta}\; J_{\alpha-\beta+q+k}\,(xy)\,dx\,dy\,.$$

Now, the proof is similar to the proof of Theorem 4.1.1 of Lee [3].

Thus, we can write:

$(-1)^{k'+q'}\; \xi^k\; D_\xi^q\; t^{k'}\,(t^{-1}\,D_t)^{q'}\; t^{2\beta-1}\;\Phi\,(\xi,t)$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} y^{4\alpha+k'+2q'}\; x^k\; D_x^q\; y^{k'}\;\left(y^{-1}D_y\right)^{q'} y^{2\beta-1}\,\phi(x,y)\; e^{i\xi x}\;(yt)^{-\alpha+\beta-q'} J_{\alpha-\beta+q'+k'}(yt)\;dxdy$$

Now, we assume that ν, ν' are positive integers such that $\nu' \geq 4\alpha,\ set\ n = \nu + 2q + k$ and use the fact that $\left|z^{-\alpha+\beta-q'} J_{\alpha-\beta+q'+k'}(z)\right| \leq C$, and the estimate from [4, p.107] and the Theorem 3.11.1 from [4, p.107],

$\left|\xi^k\; D_\xi^q\;\Phi\,(\xi,t)\right| \;\leq C\; A^q\;(1+\delta)^{q+2}\;(2\beta)^k a_q\; b_k\; for\; all\; k,q\; \geq \nu,$ we obtain the following estimate.

$\left|(-1)^{k'+q'}\,\xi^k D_\xi^q t^{k'}\,(t^{-1}D_t)^{q'}\;\; t^{2\beta-1}\Phi(\xi,t)\right|$

$\leq C\; A^q\;(1+\delta)^{q+2}\;(2\beta)^k\; a_q\; b_k\; C_1^{\alpha,\beta}\;(B'+\rho')^{q'}\; q_q'$

$\times\;\left[(A'+\rho')^{k'+2q'}\; a_{k+2q}' +\;(A'+\delta')^{n+2}\; a_{n+2}\right]$

$\leq C\; A^q\;(1+\delta)^{q+2}\;(2\beta)^k\; a_q\; b_k\; C_1^{\alpha,\beta}(B'+\rho')^{q'}\; q_q'\;\left(H_1'(A'+\delta')\right)^{k'+2q'}\; a_k' + 2q'$

$\times\;\left[1+R_q'(A+\delta)^{k'}\; H_q'^{\nu+2q'}\; a_{\nu'+2q'}\right]$

$\leq C\; A^q\;(1+\delta)^{q+2}\;(2B)^k\; a_q b_k\; C_2^{\alpha,\beta}(B'+\rho')^{q'}\; q_q'\;\left(H_1'(A'+\delta')\right)^{k'+2q'}\; a_{k'+2q'}$

$\leq C\; A^q\;(1+\delta)^{q+2}\;(2B)^k\; a_q\; b_k\; C_2^{\alpha,\beta} R_q'^2 (A'B'H_1'^2+\rho')^{k'}\, a_{k'}\; b_{q'}\;(A'^2\; H_1'^6+\delta')^{q'}\; a_{q'}^2\,.$

Thus proof is completed.

Theorem 5.2: If $a_k, a_{k'}$ satisfy the condition $a_p\;\leq\; R_q\; H_q^p\;\; \min a_q\; a_{p-q'}\; p\;\in\;\mathbb{N}_0\,,\;\; 0 \leq q \leq p$ then for $(\alpha-\beta)\;\geq\; -\frac{1}{2}$, $F\,h_{\alpha,\beta}$ is continuous linear mapping from the space $F\,H_{\alpha,\beta,a_k,A,a_k',A'}$ into the space $F\,H_{\alpha,\beta}^{a_q,a_q^{-1},B'}$,

where $B_q' =\; A'^2\; H_q'^6$.

Proof: Following the procedure of the proof of the above theorem we get

$$\left|(-1)^{k'+q'}\xi^k D_\xi^q t^{k'}(t^{-1}D_t)^{q'}\tau^{2\beta-1}\Phi(\xi,t)\right|$$

$$\leq C\, A^q\,(1+\delta)^{q+2}\,(2B)^k\, a_q\, b_k\, C_{k'}^{\alpha,\beta}\,(A^2 H_1^6+\delta')\, q'\, a_{q'}^{2,} \;.$$

6. Generalized Fourier-Hankel transformation of the Ultra-Distributions and its Inversion:

For $(\alpha-\beta)\geq -\frac{1}{2}$, we define the generalized F-H transformation $F\,h'_{\alpha,\beta}$ on each of the dual spaces $F\,H'_{\alpha,\beta,a_k,a'_k}$, $F\,H_{\alpha,\beta,a_k,A,a'_k,A'}$, $F\,H_{\alpha,\beta}^{b_q,B,b'_q,B'}$, $F\,H_{\alpha,\beta}^{b_q,B,b'_q}$ $and\ F\,H'^{\,b_q,B,b'_q,B'}_{\alpha,\beta,a_k,A,a'_k,A'}$ as follows:

$$\langle F,\Phi\rangle = \langle f,\phi\rangle \tag{6.1}$$

where $\Phi = F\,h_{\alpha,\beta}\,\phi$, $F = f\,h'_{\alpha,\beta}\,\phi$, $\phi\in F\,H_{\alpha,\beta}$ $and\ f$ belongs to the corresponding dual space.

The generalized Fourier transform of $f\in D'$ is defined to be the element of $F\in Z'$ such that the generalized Parseval relation

$$\langle F,\psi\rangle = (2\pi)^n\,\langle f,\phi^\nu\rangle\ ,\qquad where\ f\in F\,H'_{\alpha,\beta}\ ,\qquad \phi^\nu\in F\,H_{\alpha,\beta}\ .$$

$$\Phi = f\,h_{\alpha,\beta}\,\phi\ ,\qquad F = f\,h'_{\alpha,\beta}\,f\ \ and$$

$$\Phi\in F\,H_{\alpha,\beta}\ ,\ \phi^\nu\in F\,H_{\alpha,\beta}\ .$$

Thus, $f\,h_{\alpha,\beta}$ on $F\,H'_{\alpha,\beta}$ is the adjoint of the mapping $\Phi\longrightarrow(2\pi)^n\,\phi^\nu$.

Since $f^{-1}[f] = (2\pi)^{-n}\,[f\,(f^\nu)]$ and

$h'_{\alpha,\beta} = h^{-1}_{\alpha,\beta}$, we also have

$$\langle f\,h'_{\alpha,\beta}\,f,\Phi\rangle = (2\pi)^{-1}\,\langle f,\ f\ h'_{\alpha,\beta}\,\Phi\rangle \tag{6.2}$$

The inverse Fourier-Hankel type transform can therefore be defined as :

$$f = \left(f\,h'_{\alpha,\beta}\right)^{-1}F,\quad (\alpha-\beta)\geq -\frac{1}{2}\ .$$

Now, applying theorem due to Zemanian [6, pp. 21-23] to Theorems 5.1 and 5.2 above and in view of definition (6.2) above, we can state the following theorems.

Theorem 6.1 : Let $(\alpha-\beta)\geq -\frac{1}{2}$. If $a_k,\ a'_k$ satisfy the condition (3.5), then the generalized Fourier-Hankel type transform $f\,h'_{\alpha,\beta}$ is a continuous linear mapping from the dual space

$$F\,H'^{\,a_q,q_q{}^2,B_1^2}_{\alpha,\beta,b_k,B'}\ into\ F\,H'^{\,b_q,B,b'_q,B'}_{\alpha,\beta,a_k,A}\ ,\qquad where\ A'_1 = A'\,B'\,H_1'^2\ and\ B'_1 = A'^2\,H_1'^6\ .$$

Theorem 6.2: Let $(\alpha-\beta) \geq -\frac{1}{2}$. If a_k , $a_k^{'}$ satisfy the condition (3.5), then the generalized Fourier-Hankel type transform $f\, h_{\alpha,\beta}^{'}$ is a continuous linear mapping from the dual space $F\,H_{\alpha,\beta,b_k,B,a_k^{'},b_k^{'},A_1^{'}}^{a_q,a_q^{'2},B_1^{'}}$ *into* $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{'\,b_q,B,b_q^{'},B^{'}}$ where $A_1^{'}=A^{'}B^{'}H_1^{'2}$ *and* $B_1^{'}=A^{'2}\,H_1^{'6}$.

7. An operational calculus:

The distributional Fourier-Hankel type transform generates an operational calculus by means of which certain differential equations involving generalized functions can be solved. We now consider the differential equation:

$$P\left(D_x^k\ ,\Delta_{\alpha,\beta,y}^{k^{'}}\right)u=g \tag{7.1}$$

where $p(x,y)$ is a polynomial having no zeros on $-\infty<x,y<0$, g is a given member of $F\,H_{\alpha,\beta,a_k,a_k^{'}}^{'}$ or $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{'}$ or $F\,H_{\alpha,\beta}^{'\,b_q,B,b_q^{'},B^{'}}$ or $F\,H_{\alpha,\beta}^{'\,b_q,B,b_q^{'}}$ or $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{'\,b_q,B,b_q^{'},B^{'}}$, P is a polynomial such that

$$P\left((-i\xi),\qquad(-t^2)\right)\neq 0$$

and u is an unknown generalized function which is to be determined. Using

$$\mathcal{F}\,h_{\alpha,\beta}^{'}\ \left(\left(D_x^k\ ,\Delta_{\alpha,\beta,y}^{k^{'}}\right)f\right)=\ (-i\xi)^k\ (-t^2)^{k^{'}}\ F\,h_{\alpha,\beta}^{'}\ (f)\text{ and applying}$$

$\mathcal{F}\,h_{\alpha,\beta}^{'}$ to (7.1) , we obtain

$$\left(\,P\,(-i\xi)^k\ (-t^2)^{k^{'}}\right)U\,(\xi,t)=G\,(\xi,t) \tag{7.2}$$

where U and G are distributional Fourier-Hankel type transforms of u *and* g respectively.

Since $\left(\,P\,(-i\xi)^k\ ,\ (-t^2)^{k^{'}}\right)$ is a multiplier in $F\,H_{\alpha,\beta,a_k,a_k^{'}}$, $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}$, $F\,H_{\alpha,\beta}^{'\,b_q,B,b_q^{'},B^{'}}$, $F\,H_{\alpha,\beta}^{'\,b_q,B,b_q^{'}}$, $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{'\,b_q,B,b_q^{'},B^{'}}$, $1/P\left(-i\xi\right)^k\ ,(-t^2)^{k^{'}}\big)$ is a multiplier in the corresponding dual space for a_k, $a_k^{'}$ satisfying the condition (3.5) and b_q , $b_q^{'}$ satisfying the condition (3.1) .

Therefore,

$$U\,(\xi,t)=G(\xi,t)\ /\ P\left((-i\xi)^k\ \ ,(-t^2)^{k^{'}}\right)\ .$$

By taking the generalized inverse Fourier-Hankel type transform $\left(Fh_{\alpha,\beta}^{'}\right)^{-1}$, the solution is given by

$$u\,(x,y)=\ \left(Fh_{\alpha,\beta}^{'}\right)^{-1}F(\xi,t)/P\left((-i\xi)^k\ \ ,(-t^2)^{k^{'}}\right).$$

This means that for each testing function ϕ belonging to one of the spaces $F\,H_{\alpha,\beta,a_k,a_k^{'}}$ *or* $F\,H_{\alpha,\beta,a_k,A,a_q^{'},A^{'}}$ or $F\,H_{\alpha,\beta}^{'\,b_q,B,b_q^{'},B^{'}}$, $F\,H_{\alpha,\beta}^{'\,b_q,B,b_q^{'}}$, $F\,H_{\alpha,\beta,a_k,A,a_k^{'},A^{'}}^{'\,b_q,B,b_q^{'},B^{'}}$ P is a polynomial such that $P\left(-i\xi,-t^2\right)\neq 0$, the unknown u belonging to the corresponding dual space is given by

$$\langle u, \left(Fh'_{\alpha,\beta}\right)^{-1} \Phi\rangle = \langle F(\xi,t)/P((-i\xi)^k , (-t^2)^{k'}), \Phi(\xi,t)\rangle$$
$$= \langle F\,(\xi,t)\,\Phi\,(\xi,t)/P\left((-i\xi)^k, (-t^2)^{k'}\right)\rangle,$$

where $\Phi = \mathcal{F}\, h_{\alpha,\beta}\, \phi$.

References:

1. I.M. Gelfand and G.E.Shilov, Generalized Functions, Vol. 2, Academic press, New York, (1968).
2. H. Komatsu, J. Fac. Sci. Uni. Tokyo See. IA 20 (1973), 25-105.
3. W.Y. Lee, Siam. J. Math. Analysis, 5(2) (1974) , 336-48.
4. R.S. Pathak, Integral Transformations of Generalized Functions and their Applications, Gordon and Breach Science Publishers, Amsterdam (1997).
5. C. Roumieu, J. d' Analyse Math., 10 (1962-63), 153-92.
6. A.H. Zemanian, Generalized Integral Transformations, Interscience Publication, New York, (1968).