

Some Results on the Zeros of Polynomials

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Abstract : In the literature there exist many results on the location of zeros of polynomials. In this paper we generalise some existing results on the zeros of polynomials by restricting their coefficients to more general conditions.

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1. Introduction and Statement of Results

Recently Y. Choo [1] found expressions for the bounds of the zeros of polynomials involving only some of the coefficients under certain conditions on the coefficients. In fact , he proved the following results:

Theorem A: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with

$\operatorname{Re}(a_j) = \alpha_j$ and $\operatorname{Im}(a_j) = \beta_j$, $j = 0, 1, \dots, n$, such that for some $k_1 \geq 1$, $k_2 \geq 1$, $k_3 \geq 1$, $k_4 \geq 1$, $t > 0$ and some nonnegative integers k and s, and positive integers l and q,

$$k_1 t^{2\lceil \frac{n}{2} \rceil} \alpha_{2\lceil \frac{n}{2} \rceil} \leq \dots \leq t^{2k+2} \alpha_{2k+2} \leq t^{2k} \alpha_{2k} \geq \dots \geq t^2 \alpha_2 \geq \alpha_0,$$

$$k_2 t^{2\lceil \frac{n}{2} \rceil} \alpha_{2\lceil \frac{n}{2} \rceil - 1} \leq \dots \leq t^{2l} \alpha_{2l+1} \leq t^{2l-2} \alpha_{2l-1} \geq \dots \geq t^2 \alpha_3 \geq \alpha_1,$$

$$k_3 t^{2\lceil \frac{n}{2} \rceil} \beta_{2\lceil \frac{n}{2} \rceil} \leq \dots \leq t^{2s+2} \beta_{2s+2} \leq t^{2s} \beta_{2s} \geq \dots \geq t^2 \beta_2 \geq \beta_0,$$

$$k_4 t^{2\lceil \frac{n}{2} \rceil} \beta_{2\lceil \frac{n}{2} \rceil - 1} \leq \dots \leq t^{2q} \beta_{2q+1} \leq t^{2q-2} \beta_{2q-1} \geq \dots \geq t^2 \beta_3 \geq \beta_1.$$

If n is even, then P(z) has all its zeros in the disk $R_1 \leq |z| \leq R_2$, where

$$R_1 = \frac{t|a_0|}{M_1} \text{ and } R_2 = \frac{M_2}{t^{n-1}|a_n|},$$

with

$$M_1 = t^n |a_n| + t^{n-1} |a_{n-1}| + t |a_1| + t^n \{(k_1 - 1)\alpha_n\} + \{(k_3 - 1)\beta_n\}$$

$$\begin{aligned}
 & + t^{n-1} \left\{ (k_2 - 1) \alpha_{n-1} \right| + \left| (k_4 - 1) \beta_{n-1} \right\} + 2(t^{2k} \alpha_{2k} + t^{2l-1} \alpha_{2l-1} \\
 & + t^{2s} \beta_{2s} + t^{2q-1} \beta_{2q-1}) - t^n (k_1 \alpha_n + k_3 \beta_n) - t^{n-1} (k_2 \alpha_{n-1} + k_4 \beta_{n-1}) \\
 & - t(\alpha_1 + \beta_1) - (\alpha_0 + \beta_0),
 \end{aligned}$$

and

$$\begin{aligned}
 M_2 = & t^{n-1} |a_{n-1}| + t|a_1| + |a_0| + t^n \left\{ (k_1 - 1) \alpha_n \right| + \left| (k_3 - 1) \beta_n \right\} \\
 & + t^{n-1} \left\{ (k_2 - 1) \alpha_{n-1} \right| + \left| (k_4 - 1) \beta_{n-1} \right\} + 2(t^{2k} \alpha_{2k} + t^{2l-1} \alpha_{2l-1} \\
 & + t^{2s} \beta_{2s} + t^{2q-1} \beta_{2q-1}) - t^n (k_1 \alpha_n + k_3 \beta_n) - t^{n-1} (k_2 \alpha_{n-1} + k_4 \beta_{n-1}) \\
 & - t(\alpha_1 + \beta_1) - (\alpha_0 + \beta_0).
 \end{aligned}$$

If n is odd, then P(z) has all its zeros in the disk $R_3 \leq |z| \leq R_4$, where

$$R_3 = \frac{t|a_0|}{M_3} \text{ and } R_4 = \frac{M_4}{t^{n-1}|a_n|},$$

and M_3 and M_4 are respectively the same as M_1 and M_2 except that k_1, k_2, k_3 and k_4 are respectively replaced by k_2, k_1, k_4 and k_3 .

Theorem B: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that

$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $j = 0, 1, \dots, n$, and for some $k_1 > 0, k_2 > 0, t > 0$ and some nonnegative integer k and positive integer l,

$$\begin{aligned}
 k_1 t^{2[\frac{n}{2}]} \left| a_{2[\frac{n}{2}]} \right| \leq \dots \leq t^{2k+2} |a_{2k+2}| \leq t^{2k} |a_{2k}| \geq \dots \geq t^2 |a_2| \geq |a_0|, \\
 k_2 t^{2[\frac{n}{2}]} \left| a_{2[\frac{n}{2}]-1} \right| \leq \dots \leq t^{2l} |a_{2l+1}| \leq t^{2l-2} |a_{2l-1}| \geq \dots \geq t^2 |a_3| \geq |a_1|.
 \end{aligned}$$

If n is even, then P(z) has all its zeros in the disk $R_5 \leq |z| \leq R_6$, where

$$R_5 = \frac{t|a_0|}{M_5} \text{ and } R_6 = \frac{M_6}{t^{n-1}|a_n|},$$

with

$$\begin{aligned}
 M_5 = & t^n |a_n| + t^{n-1} |a_{n-1}| + t|a_1| + t^n \left| (k_1 - 1) a_n \right| + t^{n-1} \left| (k_2 - 1) a_{n-1} \right| \\
 & + \cos \alpha \left\{ 2t^{2k} |a_{2k}| + 2t^{2l-1} |a_{2l-1}| - t^n k_1 |a_n| - t^{n-1} k_2 |a_{n-1}| - t|a_1| - |a_0| \right\} \\
 & + \sin \alpha \left\{ k_1 t^n |a_n| + k_2 t^{n-1} |a_{n-1}| + t|a_1| + |a_0| + 2 \sum_{j=2}^{n-2} t^j |a_j| \right\},
 \end{aligned}$$

and

$$\begin{aligned}
 M_6 = & t^{n-1} |a_{n-1}| + t|a_1| + |a_0| + t^n \left| (k_1 - 1) a_n \right| + t^{n-1} \left| (k_2 - 1) a_{n-1} \right| \\
 & + \cos \alpha \left\{ 2t^{2k} |a_{2k}| + 2t^{2l-1} |a_{2l-1}| - t^n k_1 |a_n| - t^{n-1} k_2 |a_{n-1}| - t|a_1| - |a_0| \right\} \\
 & + \sin \alpha \left\{ k_1 t^n |a_n| + k_2 t^{n-1} |a_{n-1}| + t|a_1| + |a_0| + 2 \sum_{j=2}^{n-2} t^j |a_j| \right\}.
 \end{aligned}$$

If n is odd, then $P(z)$ has all its zeros in the disk $R_7 \leq |z| \leq R_8$, where

$$R_7 = \frac{t|a_0|}{M_7} \text{ and } R_8 = \frac{M_8}{t^{n-1}|a_n|},$$

and M_7 and M_8 are respectively the same as M_5 and M_6 except that k_1 and k_2 are respectively replaced by k_2 and k_1 .

The aim of this paper is to generalise Theorems A and B with less restrictive conditions on the coefficients. More precisely, we prove the following results:

Theorem 1: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j) = \alpha_j$ and $\operatorname{Im}(a_j) = \beta_j$, $j = 0, 1, \dots, n$, such that for some $k_1 \geq 1$, $k_2 \geq 1$, $k_3 \geq 1$, $k_4 \geq 1$, some nonnegative integers k and s , and positive integers l and q , and for some τ_j , $0 < \tau_j \leq 1$, $j = 1, 2, 3, 4$,

$$k_1 \alpha_{2\lfloor \frac{n}{2} \rfloor} \leq \dots \leq \alpha_{2k+2} \leq \alpha_{2k} \geq \dots \geq \alpha_2 \geq \tau_1 \alpha_0,$$

$$k_2 \alpha_{2\lfloor \frac{n}{2} \rfloor - 1} \leq \dots \leq \alpha_{2l+1} \leq \alpha_{2l-1} \geq \dots \geq \alpha_3 \geq \tau_2 \alpha_1,$$

$$k_3 \beta_{2\lfloor \frac{n}{2} \rfloor} \leq \dots \leq \beta_{2s+2} \leq \beta_{2s} \geq \dots \geq \beta_2 \geq \tau_3 \beta_0,$$

$$k_4 \beta_{2\lfloor \frac{n}{2} \rfloor - 1} \leq \dots \leq \beta_{2q+1} \leq \beta_{2q-1} \geq \dots \geq \beta_3 \geq \tau_4 \beta_1.$$

If n is even, then $P(z)$ has all its zeros in the disk $R_9 \leq |z| \leq R_{10}$, where

$$R_9 = \frac{|a_0|}{M_9} \text{ and } R_{10} = \frac{M_{10}}{|a_n|},$$

with

$$\begin{aligned} M_9 = & |a_n| + |a_{n-1}| + |a_1| + |(k_1 - 1)\alpha_n| + |(k_3 - 1)\beta_n| \\ & + |(k_2 - 1)\alpha_{n-1}| + |(k_4 - 1)\beta_{n-1}| + 2(\alpha_{2k} + \alpha_{2l-1} \\ & + \beta_{2s} + \beta_{2q-1}) - (k_1 \alpha_n + k_3 \beta_n) - (k_2 \alpha_{n-1} + k_4 \beta_{n-1}) \\ & - (\tau_2 \alpha_1 + \tau_4 \beta_1) - (\tau_1 \alpha_0 + \tau_3 \beta_0) \\ & + |(\tau_1 - 1)\alpha_0| + |(\tau_2 - 1)\alpha_1| + |(\tau_3 - 1)\beta_0| + |(\tau_4 - 1)\beta_1| \end{aligned}$$

and

$$\begin{aligned} M_{10} = & |a_{n-1}| + |a_1| + |a_0| + |(k_1 - 1)\alpha_n| + |(k_3 - 1)\beta_n| \\ & + |(k_2 - 1)\alpha_{n-1}| + |(k_4 - 1)\beta_{n-1}| + 2(\alpha_{2k} + \alpha_{2l-1} \\ & + \beta_{2s} + \beta_{2q-1}) - (k_1 \alpha_n + k_3 \beta_n) - (k_2 \alpha_{n-1} + k_4 \beta_{n-1}) \\ & - (\tau_2 \alpha_1 + \tau_4 \beta_1) - (\tau_1 \alpha_0 + \tau_3 \beta_0) + |(\tau_1 - 1)\alpha_0| + |(\tau_2 - 1)\alpha_1| \\ & + |(\tau_3 - 1)\beta_0| + |(\tau_4 - 1)\beta_1| \end{aligned}$$

If n is odd, then P(z) has all its zeros in the disk $R_{11} \leq |z| \leq R_{12}$, where

$$R_{11} = \frac{|a_0|}{M_{11}} \text{ and } R_{12} = \frac{M_{12}}{|a_n|},$$

and M_{11} and M_{12} are respectively the same as M_9 and M_{10} except that k_1, k_2, k_3 and k_4 are respectively replaced by k_2, k_1, k_4 and k_3 .

Applying Theorem 1 to the polynomial P(t z), we get the following result:

Corollary 1: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n with

$\operatorname{Re}(a_j) = \alpha_j$ and $\operatorname{Im}(a_j) = \beta_j, j = 0, 1, \dots, n$, such that for some $k_1 \geq 1, k_2 \geq 1, k_3 \geq 1, k_4 \geq 1$, some nonnegative integers k and s, and positive integers l and q, and for some $\tau_j, 0 < \tau_j \leq 1, j = 1, 2, 3, 4$,

$$k_1 t^{2[\frac{n}{2}]} \alpha_{2[\frac{n}{2}]} \leq \dots \leq t^{2k+2} \alpha_{2k+2} \leq t^{2k} \alpha_{2k} \geq \dots \geq t^2 \alpha_2 \geq \tau_1 \alpha_0,$$

$$k_2 t^{2[\frac{n}{2}]} \alpha_{2[\frac{n}{2}-1]} \leq \dots \leq t^{2l} \alpha_{2l+1} \leq t^{2l-2} \alpha_{2l-1} \geq \dots \geq t^2 \alpha_3 \geq \tau_2 \alpha_1,$$

$$k_3 t^{2[\frac{n}{2}]} \beta_{2[\frac{n}{2}]} \leq \dots \leq t^{2s+2} \beta_{2s+2} \leq t^{2s} \beta_{2s} \geq \dots \geq t^2 \beta_2 \geq \tau_3 \beta_0,$$

$$k_4 t^{2[\frac{n}{2}]} \beta_{2[\frac{n}{2}-1]} \leq \dots \leq t^{2q} \beta_{2q+1} \leq t^{2q-2} \beta_{2q-1} \geq \dots \geq t^2 \beta_3 \geq \tau_4 \beta_1.$$

If n is even, then P(z) has all its zeros in the disk $R_{13} \leq |z| \leq R_{14}$, where

$$R_{13} = \frac{t|a_0|}{M_{13}} \text{ and } R_{14} = \frac{M_{14}}{t^{n-1}|a_n|},$$

with

$$\begin{aligned} M_{13} = & t^n |a_n| + t^{n-1} |a_{n-1}| + t |a_1| + t^n \{ (k_1 - 1) \alpha_n \} + \{ (k_3 - 1) \beta_n \} \\ & + t^{n-1} \{ (k_2 - 1) \alpha_{n-1} \} + \{ (k_4 - 1) \beta_{n-1} \} + 2(t^{2k} \alpha_{2k} + t^{2l-1} \alpha_{2l-1} \\ & + t^{2s} \beta_{2s} + t^{2q-1} \beta_{2q-1}) - t^n (k_1 \alpha_n + k_3 \beta_n) - t^{n-1} (k_2 \alpha_{n-1} + k_4 \beta_{n-1}) \\ & - t(\tau_2 \alpha_1 + \tau_4 \beta_1) - (\tau_1 \alpha_0 + \tau_3 \beta_0) \\ & + |(\tau_1 - 1) \alpha_0| + |(\tau_2 - 1) \alpha_1| + |(\tau_3 - 1) \beta_0| + |(\tau_4 - 1) \beta_1| \end{aligned}$$

and

$$\begin{aligned} M_{14} = & t^{n-1} |a_{n-1}| + t |a_1| + |a_0| + t^n \{ (k_1 - 1) \alpha_n \} + \{ (k_3 - 1) \beta_n \} \\ & + t^{n-1} \{ (k_2 - 1) \alpha_{n-1} \} + \{ (k_4 - 1) \beta_{n-1} \} + 2(t^{2k} \alpha_{2k} + t^{2l-1} \alpha_{2l-1} \\ & + t^{2s} \beta_{2s} + t^{2q-1} \beta_{2q-1}) - t^n (k_1 \alpha_n + k_3 \beta_n) - t^{n-1} (k_2 \alpha_{n-1} + k_4 \beta_{n-1}) \\ & - t(\tau_2 \alpha_1 + \tau_4 \beta_1) - (\tau_1 \alpha_0 + \tau_3 \beta_0) + |(\tau_1 - 1) \alpha_0| + |(\tau_2 - 1) \alpha_1| \\ & + |(\tau_3 - 1) \beta_0| + |(\tau_4 - 1) \beta_1| \end{aligned}$$

If n is odd, then P(z) has all its zeros in the disk $R_{15} \leq |z| \leq R_{16}$, where

$$R_{15} = \frac{t|a_0|}{M_{15}} \text{ and } R_{16} = \frac{M_{16}}{t^{n-1}|a_n|},$$

and M_{15} and M_{16} are respectively the same as M_{13} and M_{14} except that k_1, k_2, k_3 and k_4 are respectively replaced by k_2, k_1, k_4 and k_3 .

Remark 1: Taking $\tau_j = 1, j=1,2,3,4$ in Corollary 1, we get Theorem A.

Theorem 2: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that

$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 0, 1, \dots, n$, and for some $k_1 > 0, k_2 > 0$, some nonnegative integer k and positive integer l, and some $\tau_j, 0 < \tau_j \leq 1, j = 1, 2$,

$$\begin{aligned} k_1 \left| a_{2\lceil \frac{n}{2} \rceil} \right| &\leq \dots \leq |a_{2k+2}| \leq |a_{2k}| \geq \dots \geq |a_2| \geq \tau_1 |a_0|, \\ k_2 \left| a_{2\lfloor \frac{n}{2} \rfloor - 1} \right| &\leq \dots \leq |a_{2l+1}| \leq |a_{2l-1}| \geq \dots \geq |a_3| \geq \tau_2 |a_1|. \end{aligned}$$

If n is even, then P(z) has all its zeros in the disk $R_{17} \leq |z| \leq R_{18}$, where

$$R_{17} = \frac{|a_0|}{M_{17}} \text{ and } R_{18} = \frac{M_{18}}{|a_n|},$$

with

$$\begin{aligned} M_{17} = & |a_n| + |a_{n-1}| + |a_1| + |(k_1 - 1)a_n| + |(k_2 - 1)a_{n-1}| + |(\tau_1 - 1)a_0| + |(\tau_2 - 1)a_1| \\ & + \cos \alpha \{ 2|a_{2k}| + 2|a_{2l-1}| - k_1 |a_n| - k_2 |a_{n-1}| - |\tau_2 a_1| - |\tau_1 a_0| \} \\ & + \sin \alpha \left\{ k_1 |a_n| + k_2 |a_{n-1}| + |\tau_2 a_1| + |\tau_1 a_0| + 2 \sum_{j=2}^{n-2} |a_j| \right\}, \end{aligned}$$

and

$$\begin{aligned} M_{18} = & |a_{n-1}| + |a_1| + |a_0| + |(k_1 - 1)a_n| + |(k_2 - 1)a_{n-1}| + |(\tau_1 - 1)a_0| + |(\tau_2 - 1)a_1| \\ & + \cos \alpha \{ 2|a_{2k}| + 2|a_{2l-1}| - k_1 |a_n| - k_2 |a_{n-1}| - |\tau_2 a_1| - |\tau_1 a_0| \} \\ & + \sin \alpha \left\{ k_1 |a_n| + k_2 |a_{n-1}| + |\tau_2 a_1| + |\tau_1 a_0| + 2 \sum_{j=2}^{n-2} |a_j| \right\}. \end{aligned}$$

If n is odd, then P(z) has all its zeros in the disk $R_{19} \leq |z| \leq R_{20}$, where

$$R_{19} = \frac{|a_0|}{M_{19}} \text{ and } R_{20} = \frac{M_{20}}{|a_n|},$$

and M_{19} and M_{20} are respectively the same as M_{17} and M_{18} except that k_1 and k_2 are respectively replaced by k_2 and k_1 .

Applying Theorem 2 to the polynomial $P(tz)$, we get the following result:

Corollary 2: Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n such that

$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $j = 0, 1, \dots, n$, and for some $k_1 > 0, k_2 > 0, t > 0$, some nonnegative integer k and positive integer l, and some $\tau_j, 0 < \tau_j \leq 1, j = 1, 2$,

$$k_1 t^{2\lceil \frac{n}{2} \rceil} |a_{2\lceil \frac{n}{2} \rceil}| \leq \dots \leq t^{2k+2} |a_{2k+2}| \leq t^{2k} |a_{2k}| \geq \dots \geq t^2 |a_2| \geq \tau_1 |a_0|,$$

$$k_2 t^{2\lceil \frac{n}{2} \rceil} |a_{2\lceil \frac{n}{2} \rceil - 1}| \leq \dots \leq t^{2l} |a_{2l+1}| \leq t^{2l-2} |a_{2l-1}| \geq \dots \geq t^2 |a_3| \geq \tau_2 |a_1|.$$

If n is even, then $P(z)$ has all its zeros in the disk $R_{21} \leq |z| \leq R_{22}$, where

$$R_{21} = \frac{t|a_0|}{M_{21}} \text{ and } R_{22} = \frac{M_{22}}{t^{n-1} |a_n|},$$

with

$$M_{21} = t^n |a_n| + t^{n-1} |a_{n-1}| + t |a_1| + t^n |(k_1 - 1)a_n| + t^{n-1} |(k_2 - 1)a_{n-1}| + t |(\tau_2 - 1)a_1|$$

$$\begin{aligned} &+ |(\tau_1 - 1)a_0| + \cos \alpha \{2t^{2k} |a_{2k}| + 2t^{2l-1} |a_{2l-1}| - t^n k_1 |a_n| - t^{n-1} k_2 |a_{n-1}| - t |\tau_2 a_1| - |\tau_1 a_0|\} \\ &+ \sin \alpha \left\{ k_1 t^n |a_n| + k_2 t^{n-1} |a_{n-1}| + t |\tau_2 a_1| + |\tau_1 a_0| + 2 \sum_{j=2}^{n-2} t^j |a_j| \right\}, \end{aligned}$$

and

$$M_{22} = t^{n-1} |a_{n-1}| + t |a_1| + |a_0| + t^n |(k_1 - 1)a_n| + t^{n-1} |(k_2 - 1)a_{n-1}| + t |(\tau_2 - 1)a_1|$$

$$\begin{aligned} &+ |(\tau_1 - 1)a_0| + \cos \alpha \{2t^{2k} |a_{2k}| + 2t^{2l-1} |a_{2l-1}| - t^n k_1 |a_n| - t^{n-1} k_2 |a_{n-1}| - t |\tau_2 a_1| - |\tau_1 a_0|\} \\ &+ \sin \alpha \left\{ k_1 t^n |a_n| + k_2 t^{n-1} |a_{n-1}| + t |\tau_2 a_1| + |\tau_1 a_0| + 2 \sum_{j=2}^{n-2} t^j |a_j| \right\}. \end{aligned}$$

If n is odd, then $P(z)$ has all its zeros in the disk $R_{23} \leq |z| \leq R_{24}$, where

$$R_{23} = \frac{t|a_0|}{M_{23}} \text{ and } R_{24} = \frac{M_{24}}{t^{n-1} |a_n|},$$

and M_{23} and M_{24} are respectively the same as M_{21} and M_{22} except that k_1 and k_2 are respectively replaced by k_2 and k_1 .

Remark 2: Taking $\tau_1 = \tau_2 = 1$ in Corollary 2, it reduces to Theorem B.

2.Lemma

For the proof of the above results we shall make use of the following lemma:

Lemma 1: Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial of degree n with complex coefficients such that for some real β ,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 0, 1, \dots, n$$

and

$$|a_j| \geq |a_{j-1}|, j = 1, 2, \dots, n.$$

Then for any $t > 0$,

$$|ta_j - a_{j-1}| \leq (t|a_j| - |a_{j-1}|) \cos \alpha + (t|a_j| + |a_{j-1}|) \sin \alpha.$$

The proof of Lemma 1 follows from a lemma due to Govil and Rahman [2].

3. Proofs of Theorems

Proof of Theorem 1. We first consider the case when n is even. We are given that

$$k_1 \alpha_{2[\frac{n}{2}]} \leq \dots \leq \alpha_{2k+2} \leq \alpha_{2k} \geq \dots \geq \alpha_2 \geq \tau_1 \alpha_0,$$

$$k_2 \alpha_{2[\frac{n}{2}-1]} \leq \dots \leq \alpha_{2l+1} \leq \alpha_{2l-1} \geq \dots \geq \alpha_3 \geq \tau_2 \alpha_1,$$

$$k_3 \beta_{2[\frac{n}{2}]} \leq \dots \leq \beta_{2s+2} \leq \beta_{2s} \geq \dots \geq \beta_2 \geq \tau_3 \beta_0,$$

$$k_4 \beta_{2[\frac{n}{2}-1]} \leq \dots \leq \beta_{2q+1} \leq \beta_{2q-1} \geq \dots \geq \beta_3 \geq \tau_4 \beta_1.$$

For the outer bound, consider the polynomial

$$\begin{aligned} F(z) &= (1-z^2)P(z) = (1-z^2)(a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0) \\ &= -a_n z^{n+2} - a_{n-1} z^{n+1} + (a_n - a_{n-2}) z^n + (a_{n-1} - a_{n-3}) z^{n-1} + \dots \\ &\quad + (a_3 - a_1) z^3 + (a_2 - a_0) z^2 + a_1 z + a_0 \\ &= -a_n z^{n+2} - a_{n-1} z^{n+1} - (k_1 - 1) \alpha_n z^n + (k_1 \alpha_n - \alpha_{n-2}) z^n - (k_2 - 1) \alpha_{n-1} z^{n-1} \\ &\quad + (k_2 \alpha_{n-1} - \alpha_{n-3}) z^{n-1} + \sum_{j=2}^{n-4} (\alpha_{n-j} - \alpha_{n-j-2}) z^{n-j} + (\alpha_3 - \tau_2 \alpha_1) z^3 \\ &\quad + (\tau_2 \alpha_1 - \alpha_1) z^3 + (\alpha_2 - \tau_1 \alpha_0) z^2 + (\tau_1 \alpha_0 - \alpha_0) z^2 \\ &\quad + i \left\{ -(k_3 - 1) \beta_n z^n + (k_3 \beta_n - \beta_{n-2}) z^n - (k_4 - 1) \beta_{n-1} z^{n-1} + (k_4 \beta_{n-1} - \beta_{n-3}) z^{n-1} \right. \\ &\quad \left. + \sum_{j=2}^{n-4} (\beta_{n-j} - \beta_{n-j-2}) z^{n-j} + (\beta_3 - \tau_4 \beta_1) z^3 + (\tau_4 \beta_1 - \beta_1) z^3 \right. \\ &\quad \left. + (\beta_2 - \tau_3 \beta_0) z^2 + (\tau_3 \beta_0 - \beta_0) z^2 \right\} + a_1 z + a_0. \end{aligned}$$

For $|z| > 1$,

$$|F(z)| \geq |a_n| |z|^{n+2} - |z|^{n+1} \left[|a_{n-1}| + \frac{|a_1|}{|z|^n} + \frac{|a_0|}{|z|^{n+1}} + \frac{|(k_1 - 1)\alpha_n|}{|z|} + \frac{|(k_2 - 1)\alpha_{n-1}|}{|z|^2} \right]$$

$$\begin{aligned}
& + \frac{|(k_3 - 1)\beta_n|}{|z|} + \frac{|(k_4 - 1)\beta_{n-1}|}{|z|^2} + \frac{|\alpha_3 - \tau_2\alpha_1| + |(\tau_2 - 1)\alpha_1|}{|z|^{n-2}} + \frac{|\alpha_2 - \tau_1\alpha_0| + |(\tau_1 - 1)\alpha_0|}{|z|^{n-1}} \\
& + \frac{(\alpha_{n-2} - k_1\alpha_n)}{|z|} + \sum_{j=2, j \text{ even}}^{n-2k-2} \frac{(\alpha_{n-j-2} - \alpha_{n-j})}{|z|^{j+1}} \\
& + \sum_{j=n-2k, j \text{ even}}^{n-4} \frac{(\alpha_{n-j} - \alpha_{n-j-2})}{|z|^{j+1}} + \frac{(\alpha_{n-3} - k_2\alpha_{n-1})}{|z|^2} + \sum_{j=3, j \text{ odd}}^{n-2l-1} \frac{(\alpha_{n-j-2} - \alpha_{n-j})}{|z|^{j+1}} \\
& + \sum_{j=n-2l+1, j \text{ odd}}^{n-4} \frac{(\alpha_{n-j} - \alpha_{n-j-2})}{|z|^{j+1}} + \frac{(\beta_{n-2} - k_3\beta_n)}{|z|} + \sum_{j=2, j \text{ even}}^{n-2s-2} \frac{(\beta_{n-j-2} - \beta_{n-j})}{|z|^{j+1}} \\
& + \sum_{j=n-2s, j \text{ even}}^{n-4} \frac{(\beta_{n-j} - \beta_{n-j-2})}{|z|^{j+1}} + \frac{(\beta_{n-3} - k_4\beta_{n-1})}{|z|^2} + \sum_{j=3, j \text{ odd}}^{n-2q-1} \frac{(\beta_{n-j-2} - \beta_{n-j})}{|z|^{j+1}} \\
& + \sum_{j=n-2q+1, j \text{ odd}}^{n-4} \frac{(\beta_{n-j} - \beta_{n-j-2})}{|z|^{j+1}} + \frac{|\beta_3 - \tau_4\beta_1| + |(\tau_4 - 1)\beta_1|}{|z|^{n-2}} + \frac{|\beta_2 - \tau_3\beta_0| + |(\tau_3 - 1)\beta_0|}{|z|^{n-1}} \Big] \\
& \geq |z|^{n+1} (|a_n||z| - M_{10}),
\end{aligned}$$

where

$$\begin{aligned}
M_{10} = & |a_{n-1}| + |a_1| + |a_0| + |(k_1 - 1)\alpha_n| + |(k_3 - 1)\beta_n| \\
& + |(k_2 - 1)\alpha_{n-1}| + |(k_4 - 1)\beta_{n-1}| + 2(\alpha_{2k} + \alpha_{2l-1} \\
& + \beta_{2s} + \beta_{2q-1}) - (k_1\alpha_n + k_3\beta_n) - (k_2\alpha_{n-1} + k_4\beta_{n-1}) \\
& - (\tau_2\alpha_1 + \tau_4\beta_1) - (\tau_1\alpha_0 + \tau_3\beta_0) + |(\tau_1 - 1)\alpha_0| + |(\tau_2 - 1)\alpha_1| \\
& + |(\tau_3 - 1)\beta_0| + |(\tau_4 - 1)\beta_1|.
\end{aligned}$$

Thus $|F(z)| > 0$ if $|z| > \frac{M_{10}}{|a_n|} = R_{10}$. This shows that all the zeros of $F(z)$ and

hence $P(z)$ whose modulus is greater than 1 lie in $|z| \leq R_{10}$. It can be shown that $M_{10} \geq |a_n|$. Hence it follows that all the zeros of $P(z)$ with modulus less than or equal to 1 already lie in $|z| \leq R_{10}$.

For the inner bound, we consider the function

$$\begin{aligned}
F(z) &= (1 - z^2)P(z) \\
&= a_0 + Q(z),
\end{aligned}$$

where

$$\begin{aligned}
Q(z) = & -a_n z^{n+2} - a_{n-1} z^{n+1} + (a_n - a_{n-2}) z^n + (a_{n-1} - a_{n-3}) z^{n-1} + \dots \\
& + (a_3 - a_1) z^3 + (a_2 - a_0) z^2 + a_1 z \\
= & -a_n z^{n+2} - a_{n-1} z^{n+1} + a_1 z - (k_1 - 1)\alpha_n z^n + (k_1\alpha_n - \alpha_{n-2}) z^n - (k_2 - 1)\alpha_{n-1} z^{n-1}
\end{aligned}$$

$$\begin{aligned}
 & + (k_2\alpha_{n-1} - \alpha_{n-3})z^{n-1} + \sum_{j=2}^{n-4} (\alpha_{n-j} - \alpha_{n-j-2})z^{n-j} + (\alpha_3 - \tau_2\alpha_1)z^3 \\
 & + (\tau_2\alpha_1 - \alpha_1)z^3 + (\alpha_2 - \tau_1\alpha_0)z^2 + (\tau_1\alpha_0 - \alpha_0)z^2 \\
 & + i\left\{ -(k_3 - 1)\beta_n z^n + (k_3\beta_n - \beta_{n-2})z^n - (k_4 - 1)\beta_{n-1}z^{n-1} + (k_4\beta_{n-1} - \beta_{n-3})z^{n-1} \right. \\
 & \left. + \sum_{j=2}^{n-4} (\beta_{n-j} - \beta_{n-j-2})z^{n-j} + (\beta_3 - \tau_4\beta_1)z^3 + (\tau_4\beta_1 - \beta_1)z^3 \right. \\
 & \left. + (\beta_2 - \tau_3\beta_0)z^2 + (\tau_3\beta_0 - \beta_0)z^2 \right\} + a_1z.
 \end{aligned}$$

If $|z| < 1$, then

$$\begin{aligned}
 |Q(z)| & \leq |a_n| + |a_{n-1}| + |a_1| + |(k_1 - 1)\alpha_n| + |(k_2 - 1)\alpha_{n-1}| + |(k_3 - 1)\beta_n| + |(k_4 - 1)\beta_{n-1}| \\
 & + (\alpha_{n-2} - k_1\alpha_n) + \sum_{j=2, even}^{n-2k-2} (\alpha_{n-j-2} - \alpha_{n-j}) + \sum_{j=n-2k, even}^{n-4} (\alpha_{n-j} - \alpha_{n-j-2}) \\
 & + (\alpha_{n-3} - k_2\alpha_{n-1}) + \sum_{j=3, odd}^{n-2l-1} (\alpha_{n-j-2} - \alpha_{n-j}) + \sum_{j=n-2l+1, odd}^{n-4} (\alpha_{n-j} - \alpha_{n-j-2}) \\
 & + (\beta_{n-2} - k_3\beta_n) + \sum_{j=2, even}^{n-s-2} (\beta_{n-j-2} - \beta_{n-j}) + \sum_{j=n-2s, even}^{n-4} (\beta_{n-j} - \beta_{n-j-2}) \\
 & + (\beta_{n-3} - k_4\beta_{n-1}) + \sum_{j=3, odd}^{n-2q-1} (\beta_{n-j-2} - \beta_{n-j}) + \sum_{j=n-2q+1, odd}^{n-4} (\beta_{n-j} - \beta_{n-j-2}) \\
 & + (\alpha_3 - \tau_2\alpha_1) + |(\tau_2 - 1)\alpha_1| + (\alpha_2 - \tau_1\alpha_0) + |(\tau_1 - 1)\alpha_0| \\
 & + (\beta_3 - \tau_4\beta_1) + |(\tau_4 - 1)\beta_1| + (\beta_2 - \tau_3\beta_0) + |(\tau_3 - 1)\alpha\beta_0| \\
 & = M_9,
 \end{aligned}$$

where

$$\begin{aligned}
 M_9 = & |a_n| + |a_{n-1}| + |a_1| + |(k_1 - 1)\alpha_n| + |(k_3 - 1)\beta_n| \\
 & + |(k_2 - 1)\alpha_{n-1}| + |(k_4 - 1)\beta_{n-1}| + 2(\alpha_{2k} + \alpha_{2l-1} \\
 & + \beta_{2s} + \beta_{2q-1}) - (k_1\alpha_n + k_3\beta_n) - (k_2\alpha_{n-1} + k_4\beta_{n-1}) \\
 & - (\tau_2\alpha_1 + \tau_4\beta_1) - (\tau_1\alpha_0 + \tau_3\beta_0) + |(\tau_1 - 1)\alpha_0| + |(\tau_2 - 1)\alpha_1| \\
 & + |(\tau_3 - 1)\beta_0| + |(\tau_4 - 1)\beta_1|.
 \end{aligned}$$

Since $Q(0)=0$, it follows by Schwarz lemma that

$$|Q(z)| \leq M_9|z| \quad \text{for } |z| < 1.$$

Thus for $|z| < 1$

$$\begin{aligned}
 |F(z)| & \geq |a_0| - |Q(z)| \\
 & \geq |a_0| - M_9|z| \\
 & > 0,
 \end{aligned}$$

if $|z| > \frac{|a_0|}{M_9} = R_9$. Hence $F(z)$ and therefore $P(z)$ does not vanish in $|z| < R_9$.

It is easy to see that $M_9 \leq |a_0|$. Hence $P(z)$ has all its zeros in the region

$R_9 \leq |z| \leq R_{10}$ and the proof is complete in case n is even.

The proof of the theorem for the case when n is odd can be proceeded similarly.

Proof of Theorem 2. Suppose that n is even and that the coefficient conditions hold i.e.,

$$\begin{aligned} k_1|a_n| &\leq \dots \leq |a_{2k+2}| \leq |a_{2k}| \geq \dots \geq |a_2| \geq \tau_1|a_0|, \\ k_2|a_{n-1}| &\leq \dots \leq |a_{2l+1}| \leq |a_{2l-1}| \geq \dots \geq |a_3| \geq \tau_2|a_1|. \end{aligned}$$

For the outer bound, consider the polynomial

$$\begin{aligned} F(z) &= (1 - z^2)P(z) = (1 - z^2)(a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0) \\ &= -a_n z^{n+2} - a_{n-1} z^{n+1} + (a_n - a_{n-2}) z^n + (a_{n-1} - a_{n-3}) z^{n-1} + \dots \\ &\quad + (a_3 - a_1) z^3 + (a_2 - a_0) z^2 + a_1 z + a_0 \\ &= -a_n z^{n+2} - a_{n-1} z^{n+1} + a_1 z + a_0 - (k_1 - 1)a_n z^n + (k_1 a_n - a_{n-2}) z^n - (k_2 - 1)a_{n-1} z^{n-1} \\ &\quad + (k_2 a_{n-1} - a_{n-3}) z^{n-1} + \sum_{j=2}^{n-4} (a_{n-j} - a_{n-j-2}) z^{n-j} + (a_3 - \tau_2 a_1) z^3 \\ &\quad + (\tau_2 a_1 - a_1) z^3 + (a_2 - \tau_1 a_0) z^2 + (\tau_1 a_0 - a_0) z^2 \end{aligned}$$

For $|z| > 1$,

$$\begin{aligned} |F(z)| &\geq |a_n| |z|^{n+2} - |z|^{n+1} \left[|a_{n-1}| + \frac{|a_1|}{|z|^n} + \frac{|a_0|}{|z|^{n+1}} + \frac{|(k_1 - 1)a_n|}{|z|} + \frac{|(k_2 - 1)a_{n-1}|}{|z|^2} \right. \\ &\quad + \frac{|a_{n-2} - k_1 a_n|}{|z|} + \sum_{j=2, j \text{ even}}^{n-2k-2} \frac{|a_{n-j-2} - a_{n-j}|}{|z|^{j+1}} \\ &\quad + \sum_{j=n-2k, j \text{ even}}^{n-4} \frac{|a_{n-j} - a_{n-j-2}|}{|z|^{j+1}} + \frac{|a_{n-3} - k_2 a_{n-1}|}{|z|^2} \\ &\quad + \sum_{j=3, j \text{ odd}}^{n-2l-1} \frac{|a_{n-j-2} - a_{n-j}|}{|z|^{j+1}} + \sum_{j=n-2l+1, j \text{ odd}}^{n-4} \frac{|a_{n-j} - a_{n-j-2}|}{|z|^{j+1}} \\ &\quad \left. + \frac{|a_3 - \tau_2 a_1| + |(\tau_2 - 1)a_1|}{|z|^{n-2}} + \frac{|a_2 - \tau_1 a_0| + |(\tau_1 - 1)a_0|}{|z|^{n-1}} \right] \end{aligned}$$

Using Lemma 1, it can be shown that

$$|F(z)| \geq |z|^{n+1} (|a_n| |z| - M_{18}),$$

where

$$\begin{aligned} M_{18} &= |a_{n-1}| + |a_1| + |a_0| + |(k_1 - 1)a_n| + |(k_2 - 1)a_{n-1}| + |(\tau_1 - 1)a_0| + |(\tau_2 - 1)a_1| \\ &\quad + \cos \alpha \{2|a_{2k}| + 2|a_{2l-1}| - k_1|a_n| - k_2|a_{n-1}| - |\tau_2 a_1| - |\tau_1 a_0|\} \\ &\quad + \sin \alpha \left\{ k_1|a_n| + k_2|a_{n-1}| + |\tau_2 a_1| + |\tau_1 a_0| + 2 \sum_{j=2}^{n-2} |a_j| \right\}. \end{aligned}$$

Thus $|F(z)| > 0$ if $|z| > \frac{M_{18}}{|a_n|} = R_{18}$, and all the zeros of $F(z)$ and hence $P(z)$

with modulus greater than 1 lie in $|z| \leq R_{18}$. It can be shown that $M_{18} \geq |a_n|$.

Consequently all the zeros of $P(z)$ with modulus less than or equal to 1 already lie in $|z| \leq R_{18}$.

For the inner bound , we consider the function

$$\begin{aligned} F(z) &= (1 - z^2)P(z) \\ &= a_0 + Q(z), \end{aligned}$$

where

$$\begin{aligned} Q(z) &= -a_n z^{n+2} - a_{n-1} z^{n+1} + (a_n - a_{n-2}) z^n + (a_{n-1} - a_{n-3}) z^{n-1} + \dots \\ &\quad + (a_3 - a_1) z^3 + (a_2 - a_0) z^2 + a_1 z \\ &= -a_n z^{n+2} - a_{n-1} z^{n+1} + a_1 z - (k_1 - 1)a_n z^n + (k_1 a_n - a_{n-2}) z^n - (k_2 - 1)a_{n-1} z^{n-1} \\ &\quad + (k_2 a_{n-1} - a_{n-3}) z^{n-1} + \sum_{j=2}^{n-4} (a_{n-j} - a_{n-j-2}) z^{n-j} + (a_3 - \tau_2 a_1) z^3 \\ &\quad + (\tau_2 a_1 - a_1) z^3 + (a_2 - \tau_1 a_0) z^2 + (\tau_1 a_0 - a_0) z^2. \end{aligned}$$

If $|z| < 1$, then

$$\begin{aligned} |Q(z)| &\leq |a_n| + |a_{n-1}| + |a_1| + |(k_1 - 1)a_n| + |(k_2 - 1)a_{n-1}| \\ &\quad + |a_{n-2} - k_1 a_n| + \sum_{j=2, even}^{n-2} |a_{n-j-2} - a_{n-j}| + \sum_{j=n-2k, even}^{n-4} |a_{n-j} - a_{n-j-2}| \\ &\quad + |a_{n-3} - k_2 a_{n-1}| + \sum_{j=3, odd}^{n-2l-1} |a_{n-j-2} - a_{n-j}| + \sum_{j=n-2l+1, odd}^{n-4} |a_{n-j} - a_{n-j-2}| \\ &\quad + |a_3 - \tau_2 a_1| + |(\tau_2 - 1)a_1| + |a_2 - \tau_1 a_0| + |(\tau_1 - 1)a_0|. \end{aligned}$$

Using Lemma 1 , it is easy to see that

$$|Q(z)| \leq M_{17} \quad \text{for } |z| < 1,$$

where

$$\begin{aligned} M_{17} &= |a_n| + |a_{n-1}| + |a_1| + |(k_1 - 1)a_n| + |(k_2 - 1)a_{n-1}| + |(\tau_1 - 1)a_0| + |(\tau_2 - 1)a_1| \\ &\quad + \cos \alpha \{2|a_{2k}| + 2|a_{2l-1}| - k_1 |a_n| - k_2 |a_{n-1}| - |\tau_2 a_1| - |\tau_1 a_0|\} \\ &\quad + \sin \alpha \left\{ k_1 |a_n| + k_2 |a_{n-1}| + |\tau_2 a_1| + |\tau_1 a_0| + 2 \sum_{j=2}^{n-2} |a_j| \right\}. \end{aligned}$$

Since $Q(0)=0$, it follows by Schwarz Lemma that

$$|Q(z)| \leq M_{17} |z| \quad \text{for } |z| < 1.$$

Therefore , for $|z| < 1$,

$$\begin{aligned} |F(z)| &\geq |a_0| - |Q(z)| \\ &\geq |a_0| - M_{17} |z| \\ &> 0 \end{aligned}$$

if $|z| < \frac{|a_0|}{M_{17}} = R_{17}$. It can be shown that $M_{17} \geq |a_0|$. Hence $F(z)$ and therefore

$P(z)$ does not vanish in $|z| < R_{17}$. Consequently $P(z)$ has all its zeros in the disk $R_{17} \leq |z| \leq R_{18}$. That proves the result in case n is even. The case when n is odd can be proved similarly, and is omitted.

References

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