SOME THEOREMS ON HYPERNORMAL CURVES OF A KAEHLERIAN HYPERSURFACE

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ABSTRACT:

Saxena and Behari (1956) have studied Hypersurfaces of a Kaehlerian manifold. Singh (1967, 72) have been defined and studied Hypernormal curves of a Finsler subspace and on a Riemannian hypersurface respectively.

In this paper, we have defined and studied hypernormal curves of order h (h = 1, 2 …, n - 2) on a Kaehlerian hyper-surface. The equations representing a hypernormal curve of order I are obtained and some properties of these curves are investigated.

KEY WORDS & PHRASES: Hypernormal curves, Kaehlerian hypersurfaces, Riemannian Spaces.

2000 MSC: 53B40, 53C26, 46A13, 46M40, 53B35, 53C55.

1. INRODUCTION:

In an (n+1) dimensional complex space C_{n+1}, Bochner (1946, 47) referred to an allowable Co-ordinates system

$$(z^i, z^{\bar{i}}) \equiv (z^1, z^2, \ldots, z^n, z^{\bar{1}}, \ldots, z^{\bar{2}}, \ldots, z^{\bar{n}}),$$

As we shall use the following variations of the indices:

$$\begin{cases} i, j, k, \ldots\ldots = 1, 2, \ldots\ldots n+1 \\ \bar{i}, \bar{j}, \bar{k}, \ldots\ldots = \bar{1}, \bar{2}, \ldots\ldots \overline{n+1} \end{cases}$$

Let us introduce the metric defined by the positive definite Hermitian form Yano and Bochner (1953)

$$ds^2 = 2g_{i\bar{j}}\,(z^i, z^{\bar{i}})\, dz^i\, d^{\bar{j}}. \tag{1.1}$$

If the tensor g_{ij} also satisfies the condition Kaehler (1933)

$$\partial g_{i\bar{j}} / \partial z_{\bar{k}} = \partial g_{i\bar{k}} / \partial z_{\bar{j}} \tag{1.2}$$

Which is known as Kaehler's condition, and then the complex manifold with the metric satisfying the condition (1.2) is called a Kaehler manifold. We shall denote such a manifold by K^c_{n+1}. We shall assume the self-adjointness of the indices Bochner (1946).

In Yano (1965), it has been shown that an analytic hypersurface of a Kaehler manifold is also a Kaehler manifold.Let us consider an analytic hypersurface K_n^c of K_{n+1}^c. If $(v^\alpha, e^{\overline{\alpha}}) \equiv (u^1, u^2, \ldots u^n, v^{\overline{1}}, v^{\overline{2}} \ldots, v^{\overline{n}})$ denote the co-ordinates of a point in K_n^c, then the equation of the hypersurface K_n^c may be written in the form

$$z^i = z^i (u^\alpha); \; z^{\overline{i}} = z^{\overline{i}} (z^{\overline{\alpha}}) \tag{1.3}$$

Suppose that $g_{\alpha\overline{\beta}}$ is the fundamental metric tensor of K_n^c , then we have

$$g_{\alpha\beta} = g_{i\overline{j}} B_\alpha^i B_{\overline{\beta}}^{\overline{j}}, \tag{1.4}$$

Where

$$B_\alpha^i = \partial z^i / \partial u^\alpha; \; B_{\overline{\beta}}^{\overline{j}} = \partial z^{\overline{j}} / \partial u^{\overline{\beta}}$$

And variations of the indices,

$$\{\alpha, \beta, \gamma, \ldots = 1, 2, \ldots, n; \; \overline{\alpha}, \overline{\beta}, \overline{\gamma}, \ldots = \overline{1}, \overline{2}, \ldots \overline{n}.\}$$

Let $(N^i, N^{\overline{i}})$ be the components of unit normal vector to the hypersurface, then

$$2 g_{i\overline{j}} N^i N^{\overline{j}} = 1 \tag{1.5}$$

And

$$g_{i\overline{j}} N^i B_{\overline{\beta}}^{\overline{j}} = 0, \quad g_{i\overline{j}} N^{\overline{j}} B_\alpha^i = 0. \tag{1.6}$$

Consider a curve C: $z^i = z^i (s)$; $\overline{z}^i = \overline{z}^i (s)$ (where s is real) of $K^c{}_n$ and suppose that C is a non-geodesic and non-asymptotic curve. The components $(dz^i/ds, dz^{\overline{i}}/ds)$ and $(du^\alpha/ds, du^{\overline{\alpha}}/ds)$ of the unit tangent vectors of C with respect to $K^c{}_{n+1}$ and $K^c{}_n$ are related by

$$dz^i/ds = B_\alpha^i \, (du^\alpha/ds) \tag{1.7}$$

and it's conjugate.

If $(q^i, q^{\overline{i}})$ and $(p^\alpha, p^{\overline{\alpha}})$ are the components of the first curvature vectors with respect to K_{n+1}^c and K_n^c respectively, then we have from Sexena and Bihari (1956)

$$q^i = B_\alpha^i \, p^\alpha + K_n N^i \tag{1.8}$$

and it's conjugate.

Where the normal curvature $(K_n, \overline{K}_n)$ of the hypersurface is given by

$$K_n = \Omega_{\alpha\beta} \, (du^\alpha/ds) \, (du^\beta/ds), \quad \overline{K}_n = \Omega_{\overline{\alpha}\,\overline{\beta}} \, (du^\alpha/ds) \, (du^\beta/ds),$$

and

$$B^i_{\alpha\,;\,\beta} = \Omega_{\alpha\beta} N^i \; ; \quad B^{\overline{i}}_{\overline{\alpha}\,;\,\overline{\beta}} = \Omega_{\overline{\alpha}\,\overline{\beta}} N^{\overline{i}},$$

Where $(\Omega_{\alpha\beta}, \Omega_{\overline{\alpha}\,\overline{\beta}})$ are second fundamental tensors of the hypersurface.

Two vectors $(u^{\alpha}, u^{\overline{\alpha}})$ and $(v^{\alpha}, v^{\overline{\alpha}})$ of the hypersurfacae are said to be conjugate if the relation

$$\Omega_{\alpha\beta}\, u^{\alpha}\, v^{\beta} = 0 \tag{1.9}$$

and its conjugate holds.

The tensor derivative of the unit normal vector is given by weatherburn (1938),

$$N^{i}_{;\alpha} = -\,\Omega_{\alpha\beta}\, g^{\beta\gamma}\, B^{i}_{\gamma} \tag{1.10}$$

and it's conjugate.

2. HYPERNORMAL CURVES:

Assuming that $\delta/\delta s$ is the usual covariant differentiation along C, we have the first two Frenet's formulae

$$\delta\eta^{i}_{(0)} / \delta s = K_{(1)}\, \eta^{i}_{(1)}$$

and $$\delta\eta^{i}_{(1)} / \delta s = -\,K_{(1)}\, \eta^{i}_{(0)} + K_{(2)}\, \eta^{i}_{(2)} \tag{2.1}$$

and their conjugates, where

$$(\eta^{i}_{(0)}, \eta^{\bar{i}}_{(0)}) = (dz^{i}/ds, dz^{\bar{i}}/ds),\ (\eta^{i}_{(1)}, \eta^{\bar{i}}_{(1)}) \text{ and } (\eta^{i}_{(2)}, \eta^{\bar{i}}_{(2)}).$$

are the components of unit tangent vector, unit principal normal vector and unit binormal vector, $K_{(1)}$ and $K_{(2)}$ are the curvature of the first and second orders respectively.

Consider a congruence of curves given by unit vector field $\bar{\lambda} = (\lambda^{i}, \lambda^{\bar{i}})$ such that through each point of K^{c}_{n+1}, there passes exactly one curves of the congruence. At the point of hypersurface, we get

$$\lambda^{i} = t^{\alpha}\, B^{i}_{\alpha} + C\, N^{i} \tag{2.2}$$

and its conjugate, when t^{α} and C are parameters. Since $(\lambda^{i}, \lambda^{\bar{i}})$ represents a unit vector, we have

$$z\, g_{i\bar{j}}\, \lambda^{i}\, \lambda^{\bar{j}} = 1$$

and it follows by use of (1.5), (1.6) and (2.2) that

$$2g_{\alpha\bar{\beta}}\, t^{\alpha}\, t^{\bar{\beta}} = 1 - |C|^{2}.$$

Assuming that θ is the angle between the unit vectors $\bar{\lambda}$ and $\bar{N}$, we deduce

$$C = \cos\theta \text{ and } 2g_{\alpha\bar{\beta}}\, t^{\alpha}\, t^{\bar{\beta}} = \sin^{2}\theta = 1 - |C|^{2}. \tag{2.3}$$

A curve of the hypersurface will be called a hypernormal curve of order h (h=1, 2…, n-2) relative to the congruence $\bar{\lambda}$, if the variety spanned by the first curvature vector $(\eta^{i}_{(h+1)}, \eta^{\bar{i}}_{(h+1)})$ of order h contains $\bar{\lambda}$.

For a hypernormal curve of order 1, we have

$$\lambda^{i} = r\, q^{i} + \nu\, \eta^{i}_{(2)} \tag{2.4}$$

and its conjugate.

Equation (1.8), (1.10) and (2.1) yield

$$\delta q^i / \delta s = -k^2_{(1)}\, dz^i/ds + \{d(\log K_{(1)})/ds\}\, q^i + K_{(1)}K_{(2)}\, \eta^i_{(2)}$$
$$= (\delta p^\alpha / \delta s - K_n \Omega_{\gamma\psi}\, du^\psi/ds)\, B^i_\alpha$$
$$+ (\Omega_{\alpha\beta}\, p^\alpha du^\beta/ds + d K_n/ds)\, N^i \qquad (2.5)$$

Similarly, we may obtain the conjugate of the above.

Eliminating $\eta^i_{(2)}$ from (2.4) and (2.5) and using (1.8), (2.2) and (2.3), we get

$$t^\alpha = r\, p^\alpha + \omega\, [\delta p^\alpha/\delta s - K_n\, \Omega_{\gamma\psi}\, g^{\gamma\alpha}\, du^\psi/ds + K^2_{(1)}\, du^\alpha/ds$$
$$- \{d(\log K_{(1)})/ds\}\, p^\alpha], \qquad (2.6a)$$

$$\cos\theta = r\, K_n + \omega\, [\Omega_{\alpha\beta}\, p^\alpha\, du^\beta/ds + d K_n/ds$$
$$- K_n\, d(\log K_{(1)})/ds], \qquad (2.6b)$$

Where

$$w = v / K_{(1)} K_{(2)} \qquad (2.6c)$$

and their conjugates.

Using the first two Frenet's formulae (with respect to K^c_n)

$$p^\alpha = \delta u^\alpha / \delta s = K_{(1)}\, \xi^\alpha_{(1)}$$

and

$$\delta\, \xi^\alpha_{(1)} / \delta s = -K_{(1)}\, du^\alpha/ds + K_{(2)}\, \xi^\alpha_{(2)}$$

and their conjugate,

We deduce

$$\delta p^\alpha / \delta s = K^2_{(1)}\, du^\alpha/ds + \{d(\log K_{(1)})/ds\} p^\alpha + K_{(1)}K_{(2)}\, \xi^\alpha_{(2)} \qquad (2.7)$$

and its conjugate,

Where $(\xi^\alpha_{(2)}, \xi^{\bar\alpha}_{(2)})$ is the first binormal vector of the curve with respect to the hypersurface K^c_n.

Substituting this value of $\delta p^\alpha / \delta s$ in (2.6a), we obtain

$$t^\alpha = r\, p^\alpha + \omega\, [K^2_n\, (du^\alpha/ds) + \{d(\log(K_{(1)}/K^*_{(1)}))/ds\} p^\alpha$$
$$+ K_{(1)} K_{(2)}\, \xi^\alpha_{(2)} - Kn\, \Omega_{\gamma\psi}\, g^{\gamma\alpha}\, du^\psi/ds] \qquad (2.8)$$

and its conjugate may be obtained in the same way.

In the above relation, we have used

$$K^{*2}_{(1)} = K^2_{(1)} + K^2_n \qquad (2.9)$$

for simplification.

Since $\bar\lambda$ is a unit vector field, we get

$$1 = r^2 K^2_{(1)} + K^2_{(1)}\, K^2_{(2)}\, \omega^2 \qquad (2.10)$$

in view of (2.1), (2.8) and (2.6c).

The elimination of r and w from (2.6b), (2.8) and (2.10) will yield the equation for a hypernormal curve of the order one.

From equation (2.8), we find

$$g_{\alpha\bar{\beta}}\ t^{\alpha}\ du^{\bar{\beta}}/ds = 0, \qquad g_{\bar{\alpha}\beta}\ t^{\bar{\alpha}}\ du^{\bar{\beta}}/ds = 0,$$

Which proves the following theorem:

THEOREM (2.1): A necessary condition that a curve be hypernormal (of order 1) relative to the congruence $\bar{\lambda}$ is that the tangential component (to the hypersurface) of the vector field $\bar{\lambda}$ is orthonormal to the curve.

After define

$$\cos\alpha = \{g_{\alpha\bar{\beta}}\ t^{\alpha}\ \xi^{\bar{\beta}}_{(1)} + g_{\bar{\alpha}\beta}\ t^{\bar{\alpha}}\ \xi^{\beta}_{(1)}\}/\{2\sqrt{(g_{\alpha\bar{\beta}}\ t^{\alpha} t^{\bar{\beta}})}\ \sqrt{(}g_{\bar{\alpha}\beta}\ \xi^{\bar{\alpha}}_{(1)}\xi^{\beta}_{(1)}\}$$

and

$$\cos\phi = \{g_{\alpha\bar{\beta}}\ t^{\alpha}\ \xi^{\bar{\beta}}_{(2)} + g_{\bar{\alpha}\beta}\ t^{\bar{\alpha}}\ \xi^{\bar{\beta}}_{(2)}\} / \{ 2\sqrt{(g_{\alpha\bar{\beta}}\ t^{\alpha} t^{\bar{\beta}})}\ \sqrt{(}g_{\bar{\alpha}\beta}\ \xi^{\bar{\alpha}}_{(2)}\ \xi^{\beta}_{(2)}\}$$

and multiplying (2.8) respectively by $(2\ g_{\alpha\bar{\beta}}\ \xi^{\bar{\beta}}_{(1)})^{1/2}$ and $(2\ g_{\alpha\bar{\beta}}\ \xi^{\bar{\beta}}_{(2)})^{1/2}$ we deduce.

$$\sin\theta\cos\alpha = r\,K_{(1)} + \omega\{K_{(1)}\ d\ \log\left(K_{(1)}/K^{*}_{(1)}\right)/ds - K_n \Omega_{\alpha\beta}\ \xi^{\beta}_{(1)}\ du^{\alpha}/ds\} \qquad (2.11)$$

and

$$\sin\theta\ \cos\phi = \omega\,(K_{(1)}K_{(2)} - K_n\Omega_{\alpha\beta}\ du^{\alpha}/ds) \qquad (2.12)$$

Where we have used (2.3).

Defining $u = K_{(1)}/K_n$ and eliminating r and w from (2.6b), (2.11) and (2.12), we get

$$(\sin\theta\cos\alpha - u\cos\theta)(u\,K_{(2)} - \Omega_{\alpha\beta}\ \xi^{\beta}_{(2)}\ du^{\alpha}/ds) = \sin\theta.\cos\phi\{du/ds - (1+u^2)\Omega_{\alpha\beta}\ \xi^{\beta}_{(1)}\ du^{\alpha}/ds\} \qquad (2.13)$$

Where the relation $K^{*2}_{(1)} / K^2_n = 1 + K^2_{(1)} / K^2_n$ has been used in the simplification.

3. PARTICULARS CASES:

In this section, we shall consider the solution of (2.13) in the following two particulars cases:

Case (i): Let $(t^{\alpha}, t^{\bar{\alpha}})$ be orthonormal to the first binormal vector $(\xi^{\alpha}_{(2)}, \xi^{\bar{\alpha}}_{(2)})$ in particular, let $(t^{\alpha}, t^{\bar{\alpha}})$ be along the principal normal vector $(\xi^{\alpha}_{(1)}, \xi^{\bar{\alpha}}_{(1)})$. This implies $\cos\phi = 0$. Hence, we have either

$$K_{(1)} / K_n = \tan\theta \cos\alpha \tag{3.1}$$

Or

$$K_{(1)} K_{(2)} = \Omega_{\alpha\beta}\, \Omega_{\gamma\psi} \left(\frac{du^{\alpha}}{ds}\right)\left(\frac{du^{\beta}}{ds}\right)\left(\frac{du^{\gamma}}{ds}\right) \xi^{\psi}_{(2)} \tag{3.2}$$

and their conjugates, where we have used $K_n = \Omega_{\alpha\beta} \left(\frac{du^{\alpha}}{ds}\right) \times \left(\frac{du^{\beta}}{ds}\right)$ in the latter equation.

Case (ii): Let the congruence $\bar{\lambda}$ be along the normal vector $\bar{N}$ of the hypersurface. We have then $\cos\theta = -1$, $\sin\theta = 0$.

Since the curve is non-geodesic, i.e. $u \neq 0$, equation (2.12) reduces to (3.2). A special feature of (3.2) is in fact the product of the curvatures (with respect to the hypersurface) of orders one and two has been expressed in terms of the second fundamental tensor of the hypersurface.

Since the curve is non-geodesic and non-asymptotic, we have the following theorem from (3.2):

THEOREM (3.1): A necessary and sufficient condition that the curvature of order two (with respect to the hypersurface) of a non-geodesic and non-asymptotic hypernormal curve of order 1 (with respect to normal congruence) be zero is that the first binormal vector is conjugate with respect to its tangent vector.

REFERENCES

[1]. Bochner, S. (1946). Vector field and Ricci-curvature, bull. Amer. Math. Soc., 52, pp. 776-797.

[2]. Bochner, S. (1947). on compact complex manifolds, Journ. Ind. Math. Soc., 11, pp.1-21.

[3]. Yano, K. (1965). Differential geometry on complex and almost complex spaces, Pergamon Press.

[4]. Weatherburn, C.E. (1938). An introduction to Riemannian geometry and the tensor calculus, Cambridge.

[5]. Yano, K. & Bochner, S. (1953). Curvature and Betti numbers, Princeton University Press, pp. 117-119.

[6]. Saxena, S. C. & Behari, R. (1956). Hypersurfaces of a Kaehler manifold, Proc. Nat. Acad. Sci., India, 28A, pp. 170-180.

[7]. Singh, U.P. (1972). Hypernormal curves on a Riemannian hypersurface, Tensor N.S., 25, pp. 23 – 240.

[8]. Singh, U.P. (1967). Hypernormal curves of a Finsler subspace, Progress of Mathematics, 1, pp. 44-47.

[9]. Kaehler, E. (1933). *Ü*ber eine bemerkensrverte Hermitische Metrik, Abh, Math, Sem. Hemburg, 9, pp. 173-186.