

TWO NEW TYPES OF IRRESOLUTE FUNCTIONS IN TOPOLOGICAL SPACES

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Abstract. The purpose of this paper is to give two new types of functions called completely λ -irresolute function and completely weakly λ -irresolute function in topological spaces. We obtain their characterizations and basic properties.

Keywords: Topological spaces, λ -open sets, λ -closed sets, completely λ -irresolute functions, completely weakly λ -irresolute functions.

1. INTRODUCTION AND PRELIMINARIES

Functions and of course irresolute functions stand among the most important and most researched points in the whole Mathematical Science. Its importance is significant in various areas of Mathematics and related sciences. Maki [9] introduced the notion of Λ -sets in topological spaces. A Λ -set is a set A which is equal to its kernel, that is, to the intersection of all open super sets of A . Arenas et.al. [1] introduced and investigated the notion of λ -closed sets by involving Λ -sets and closed sets. In this paper, we have introduced and studied some type of functions in topological spaces called completely λ -irresolute functions and completely weakly λ -irresolute functions. Let A be a subset of a topological space (X, τ) . The closure and the interior of a set A is denoted by $Cl(A)$ and $Int(A)$ respectively. A subset A of a topological space (X, τ) is said to be regular open if $A = int(cl(A))$. A subset A of a topological space (X, τ) is said to be λ -closed [1] if $A = B \cap C$, where B is a Λ -set and C is a closed set of X . The complement

of λ -closed set is called λ -open [1]. A point $x \in X$ in a topological space (X, τ) is said to be λ -cluster point of A [3] if for every λ -open set U of X containing x , $A \cap U \neq \emptyset$. The set of all λ -cluster points of A is called the λ -closure of A and is denoted by $Cl_\lambda(A)$ [3]. A point $x \in X$ is said to be the λ -interior point of A if there exists a λ -open set U of X containing x such that $U \subset A$. The set of all λ -interior points of A is said to be the λ -interior of A and is denoted by $Int_\lambda(A)$. A set A is λ -closed (resp. λ -open) if and only if $Cl_\lambda(A) = A$ (resp. $Int_\lambda(A) = A$) [3]. The family of all λ -open (resp. λ -closed) sets of X is denoted by $\lambda O(X)$ (resp. $\lambda C(X)$). The family of all λ -open (resp. λ -closed) sets of a space (X, τ) containing the point $x \in X$ is denoted by $\lambda O(X, x)$ (resp. $\lambda C(X, x)$).

2. COMPLETELY λ -IRRESOLUTE FUNCTIONS

Definition 2.1: A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

- (i) strongly continuous [6] if $f^{-1}(V)$ is clopen in X for each subset V of Y .
- (ii) λ -irresolute [3] if $f^{-1}(V)$ is λ -open in X for every λ -open subset V of Y .
- (iii) Completely λ -irresolute if $f^{-1}(V)$ is regular open in X for every λ -open subset V of Y .

Remark 2.2: Clearly, every strongly continuous function is completely λ -irresolute and every completely λ -irresolute function is λ -irresolute. But the converses of the implications are not true in general as seen from the following examples.

Example 2.3: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X\}$. Then the identity function f on X is completely λ -irresolute but not strongly continuous.

Example 2.4: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then the identity function f on X is λ -irresolute but not completely λ -irresolute.

Theorem 2.5: The following statements are equivalent for a function $f : (X, \tau) \rightarrow (Y, \sigma)$:

- (i) f is completely λ -irresolute;
- (ii) $f^{-1}(F)$ is regular closed in X for every λ -closed set F of Y .

Proof: Clear.

Lemma 2.6: [7] Let S be an open subset of a topological space (X, τ) . Then the following hold:

- (i) If U is regular open in X , then so is $U \cap S$ in the subspace (S, τ_s) .
- (ii) If $B \subset S$ is regular open in (S, τ_s) , then there exists a regular open set U in (X, τ) such that $B = U \cap S$.

Theorem 2.7: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a completely λ -irresolute function and A is any open subset of X , then $f|_A : A \rightarrow Y$ is completely λ -irresolute.

Proof: Let F be a λ -open subset of Y . By hypothesis $f^{-1}(F)$ is regular open in X . Since A is open in X , it follows from the Lemma 2.6 that $(f|_A)^{-1}(F) = A \cap f^{-1}(F)$, which is regular open in A . Therefore, $f|_A$ is completely λ -irresolute.

Lemma 2.8: [2] Let Y be a preopen subset of a topological space (X, τ) . Then $Y \cap U$ is regular open in Y for each regular open subset U of X .

Theorem 2.9: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is completely λ -irresolute function and A is preopen subset of X , then $f|_A : A \rightarrow Y$ is completely λ -irresolute.

Proof: Similar to the Proof of Theorem 2.7.

Theorem 2.10: The following hold for functions $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$:

- (i) If f is completely λ -irresolute and g is λ -irresolute, then $g \circ f$ is completely λ -irresolute;
- (ii) If f is completely continuous and g is completely λ -irresolute, then $g \circ f$ is completely λ -irresolute.
- (iii) If f is completely λ -irresolute and g is λ -continuous, then $g \circ f$ is completely continuous function.

Proof: The proof of the theorem is easy and hence omitted.

Definition 2.11: A space X is said to be almost connected [4] (resp. λ -connected [3]) if there does not exist disjoint regular open (resp. λ -open) sets A and B such that $A \cup B = X$.

Theorem 2.12: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is completely λ -irresolute surjective function and X is almost connected, then Y is λ -connected.

Proof: Suppose that Y is not λ -connected. Then there exist disjoint λ -open sets A and B of Y such that $A \cup B = Y$. Since f is completely λ -irresolute surjective, $f^{-1}(A)$ and $f^{-1}(B)$ are regular open sets in X . Moreover, $f^{-1}(A) \cup f^{-1}(B) = X$, $f^{-1}(A) \neq \emptyset$ and $f^{-1}(B) \neq \emptyset$. This shows that X is not almost connected, which is a contradiction to the assumption that X is almost connected. By contradiction, Y is λ -connected.

Definition 2.13: A space X is said to be

- (i) nearly compact [10] if every regular open cover of X has a finite subcover;
- (ii) nearly countably compact [5] if every cover by regular open sets has a countable subcover;
- (iii) nearly Lindelof [4] if every cover of X by regular open sets has a countable subcover;
- (iv) λ -compact if every λ -open cover of X has a finite subcover;
- (v) countably λ -compact if every λ -open countable cover of X has a finite subcover;
- (vi) λ -Lindelof if every cover of X by λ -open sets has a countable subcover.

Theorem 2.14: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a completely λ -irresolute surjective function. Then the following statements hold:

- (i) If X is nearly compact, then Y is λ -compact;
- (ii) If X is nearly Lindelof, then Y is λ -Lindelof;
- (iii) If X is nearly countably compact, then Y is countably λ -compact.

Proof: (i) Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a completely λ -irresolute function of nearly compact space X onto a space Y . Let $\{U_\alpha : \alpha \in \Omega\}$ be any λ -open cover of Y . Then, $\{f^{-1}(U_\alpha) : \alpha \in \Omega\}$ is a regular open cover of X . Since X is nearly compact, there exists a finite subfamily, $\{f^{-1}(U_{\alpha_i}) : i = 1, 2, \dots, n\}$ of $\{f^{-1}(U_\alpha) : \alpha \in \Omega\}$ which cover X . It follows then that $\{U_{\alpha_i} : i = 1, 2, \dots, n\}$ is a finite subfamily of $\{U_\alpha : \alpha \in \Omega\}$ which cover Y . Hence, space Y is λ -compact space.

The proofs of other cases are similar.

Definition 2.15: A space (X, τ) is said to be:

- (i) S-closed [12] (resp. λ -closed compact) if every regular closed (resp. λ -closed) cover of X has a finite subcover;
- (ii) countably S-closed-compact (resp. countably λ -closed compact) if every countable cover of X by regular closed (resp. λ -closed) sets has a finite subcover;
- (iii) S-Lindelof [8] (resp. λ -Lindelof) if every cover of X by regular closed (resp. λ -closed) sets has a countable subcover.

Theorem 2.16: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a completely λ -irresolute surjective function. Then the following statements hold:

- (i) If X is S-closed, then Y is λ -closed compact;
- (ii) If X is S-Lindelof, then Y is λ -closed Lindelof;
- (iii) If X is countably S-closed, then Y is countably λ -closed compact.

Proof: It can be obtained similarly as the previous Theorem 2.14.

Definition 2.17: A topological space X is said to be:

- (i) almost normal [11] if for each closed set A and each regular closed set B such that $A \cap B = \emptyset$ there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.
- (ii) strongly λ -normal if for every pair of disjoint λ -closed subsets A and B of X , there exist disjoint λ -open sets U and V such that $A \subset U$ and $B \subset V$.

Theorem 2.18: If f is completely λ -irresolute λ -open from an almost regular space X onto a space Y , then Y is strongly λ -regular.

Proof: Let F be a λ -closed set in Y with $y \notin F$. Take $y = f(x)$. Since f is completely λ -irresolute, $f^{-1}(F)$ is regular closed and so closed set in X and $x \notin f^{-1}(F)$. By almost regularity of X , there exists disjoint open sets U and V such that $x \in U$ and $f^{-1}(F) \subset V$. We obtain that $y = f(x) \in f(U)$ and $F \subset f(V)$ such that $f(U)$ and $f(V)$ are disjoint λ -open sets. Thus, Y is strongly λ -regular.

Definition 2.19: A topological space X is said to be strongly λ -normal if for every pair of disjoint λ -closed subsets A and B of X , there exist disjoint λ -open sets U and V such that $A \subset U$ and $B \subset V$.

Theorem 2.20: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is completely λ -irresolute λ -open function from an almost normal space X onto a space Y , then Y is strongly λ -normal.

Proof: Let A and B be two disjoint λ -closed subsets in Y . Since f is completely λ -irresolute, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint regular closed and so closed sets in X . By almost normality of X , there exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. We obtain that $A \subset f(U)$ and $B \subset f(V)$ such that $f(U)$ and $f(V)$ are disjoint λ -open sets. Thus, Y is strongly λ -normal.

Definition 2.21: A topological space (X, τ) is said to be:

- (i) λ - T_1 [3] (resp. r - T_1 [4]) if for each pair of distinct points x and y of X , there exist λ -open (resp. regular open) sets U_1 and U_2 such that $x \in U_1$ and $y \in U_2$, $x \notin U_2$ and $y \notin U_1$.
- (ii) λ - T_2 [3] (resp. r - T_2 [4]) for each pair of distinct points x and y in X , there exist disjoint λ -open (resp. regular open) sets A and B in X such that $x \in A$ and $y \in B$.

Theorem 2.22: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is completely λ -irresolute injective function and Y is λ - T_1 , then X is r - T_1 .

Proof: Suppose that Y is λ - T_1 . For any two distinct points x and y of X , there exist λ -open sets F_1 and F_2 in Y such that $f(x) \in F_1$, $f(y) \in F_2$, $f(x) \notin F_2$ and $f(y) \notin F_1$. Since f is injective completely λ -irresolute function, we have X is r - T_1 .

Theorem 2.23: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is completely λ -irresolute injective function and Y is λ - T_2 , then X is r - T_2 .

Proof: Similar to the proof of Theorem 2.22.

Theorem 2.24: Let Y be a λ - T_2 space. Then we have the following

- (i) If $f, g : X \rightarrow Y$ are completely λ -irresolute functions, then the set $A = \{x \in X : f(x) = g(x)\}$ is λ -closed in X ;

- (i) If $f : X \rightarrow Y$ is a completely λ -irresolute function, then the set $B = \{(x, y) \in X \times X : f(x) = f(y)\}$ is λ -closed in $X \times X$.

Proof: (i): Let $x \notin A$, then $f(x) \neq g(x)$. Since Y is λ - T_2 , there exist disjoint λ -open sets U_1 and U_2 in Y such that $f(x) \in U_1$ and $g(x) \in U_2$. Since f and g are completely λ -irresolute, $f^{-1}(U_1)$ and $g^{-1}(U_2)$ are regular open sets. Put $U = f^{-1}(U_1) \cap g^{-1}(U_2)$. Then U is a regular open set containing x and $U \cap A \neq \emptyset$. Hence, we have $x \notin Cl_\lambda(A)$. This completes the proof.

(ii) Follows from (i).

3. COMPLETELY WEAKLY λ -IRRESOLUTE FUNCTIONS

Definition 3.1: A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be completely weakly λ -irresolute if for each $x \in X$ and for any λ -open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset V$.

It is obvious that every completely λ -irresolute function is completely weakly λ -irresolute and every completely weakly λ -irresolute function is λ -irresolute. However, the converse may not be true in general as shown in the following example.

Example 3.2: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Clearly the identity function $f : (X, \tau) \rightarrow (Y, \sigma)$ is completely weakly λ -irresolute but not completely λ -irresolute. Also the function f defined in example 2.4 is λ -irresolute but not completely weakly λ -irresolute.

Theorem 3.3: For a function $f : (X, \tau) \rightarrow (Y, \sigma)$, the following statements are equivalent:

- (i) f is completely weakly λ -irresolute;
- (ii) for each $x \in X$ and each λ -open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subset V$;
- (iii) $f(Cl(A)) \subseteq Cl_\lambda(f(A))$ for every subset A of X ;
- (iv) $Cl(f^{-1}(B)) \subseteq f^{-1}(Cl_\lambda(B))$ for every subset B of Y ;

- (v) for each λ -closed set V in Y , $f^{-1}(V)$ is closed in X ;
- (vi) $f^{-1}(\text{Int}_{\lambda}(B)) \subset \text{Int}(f^{-1}(B))$ for every subset B of Y .

Proof: Clear.

Theorem 3.4: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be any two functions. Then

- (i) If f is completely weakly λ -irresolute and g is λ -irresolute, then $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is completely weakly λ -irresolute;
- (ii) If f is completely continuous and g is completely weakly λ -irresolute, then $g \circ f$ is completely λ -irresolute;
- (iii) If f is strongly continuous and g is completely λ -irresolute, then $g \circ f$ is completely λ -irresolute;
- (iv) If f and g are completely λ -irresolute, then $g \circ f$ is completely λ -irresolute;
- (v) If f is completely λ -irresolute and g is completely weakly λ -irresolute, then $g \circ f$ is completely λ -irresolute;
- (vi) If f is completely weakly λ -irresolute and g is λ -continuous, then $g \circ f$ is continuous;
- (vii) If f is λ -continuous and g is completely weakly λ -irresolute, then $g \circ f$ is λ -irresolute.
- (viii) If f is continuous and g is completely weakly λ -irresolute, then $g \circ f$ is completely weakly λ -irresolute.

Proof: Follows from their respective definitions.

Recall that a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be almost open if $f^{-1}(V)$ is regular open in X for every open set V of Y .

Theorem 3.5: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is almost open and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is any function such that $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is completely λ -irresolute, then g is completely weakly λ -irresolute.

Proof: Let V be a λ -open set in (Z, η) . Since $g \circ f$ is completely λ -irresolute, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is regular open in (X, τ) . Since f is almost open surjection, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is open in Y . Therefore, g is completely weakly λ -irresolute.

Theorem 3.6: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is open surjection and $g : (Y, \sigma) \rightarrow (Z, \eta)$ is any function such that $g \circ f : (X, \tau) \rightarrow (Z, \eta)$ is completely weakly λ -irresolute, then g is completely weakly λ -irresolute.

Proof: Similar to proof of Theorem 3.5.

Theorem 3.7: If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is completely weakly λ -irresolute, then for each point $x \in X$ and each filterbase \mathcal{F} in X converging to x , the filterbase $f(\mathcal{F})$ is λ -convergent to $f(x)$.

Proof: Let $x \in X$ and be any filterbase \mathcal{F} in X converging to x . Since f is completely weakly λ -irresolute, then for any λ -open set V of (Y, σ) containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subset V$. Since \mathcal{F} is converging to x , there exists $B \in \mathcal{F}$ such that $B \subset U$. This means that $f(B) \subset V$ and hence the filterbase $f(\mathcal{F})$ is λ -convergent to $f(x)$.

Definition 3.8: A graph $G(f)$ of a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be contra λ -closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist open set U of X containing x and $V \in \lambda C(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 3.9: A graph $G(f)$ of a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra λ -closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist open set U of X containing x and $V \in \lambda C(Y, y)$ such that $f(U) \cap V = \emptyset$.

Proof: It is an immediate consequence of definition 3.8.

Theorem 3.10: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a completely weakly λ -irresolute function and (Y, σ) is a λ - T_1 space, then $G(f)$ is contra λ -closed.

Proof: Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is λ - T_1 , there exists a λ -open set V in Y such that $f(x) \in V$ and $y \notin V$. Since f is completely weakly λ -irresolute, there exists open set U of X containing x such that $f(U) \subset V$. Therefore, $f(U) \cap (Y - V) = \emptyset$ and $Y - V$ is a λ -closed subset of Y containing y . This shows that $G(f)$ is contra λ -closed.

Theorem 3.11: Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a completely weakly λ -irresolute surjective function. Then the following statements hold:

- (i) If X is Lindelof, then Y is λ -Lindelof;
- (ii) If X is countably compact, then Y is countably λ -compact.

Proof: (i): Let $\{V_\alpha : \alpha \in I\}$ be a λ -open cover of Y . Since f is completely weakly λ -irresolute, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is an open cover of X . Since X is Lindelof, there exists a countable subset I_0 of I such that $X = \bigcup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Thus, $Y = \bigcup \{(V_\alpha) : \alpha \in I_0\}$ and hence Y is λ -Lindelof.

(ii): Similar to (i).

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