CONSTRUCTION OF SUITABLE LYAPUNOV FUNCTIONS FOR SYSTEMS OF LINEAR AND NON-LINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS AND STABILITY IN THE LARGE OF THEIR TRIVIAL SOLUTIONS

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Abstract

In this paper, suitable Lyapunov functions are constructed for systems of linear and non-linear second order ordinary differential equations. Based on the definitions and results, it was shown that the trivial solution of the linear system is stable (in the sense of Lyapunov) while the trivial solution of the non-linear system is asymptotically stable.

Key Words: Lyapunov functions, Stability, trivial solutions, linear and non-linear differential equations.

1. Introduction

In the theory of ordinary differential equations, Lyapunov functions are scalar functions that are used to prove the stability of an equilibrium of ordinary differential equations. For many classes of ordinary differential equations, the existence of Lyapunov functions is a necessary and sufficient condition for stability.

Andreev [1], considered the problem of asymptotic stability of non-autonomous functional differential equations under the supposition that the derivative of Lyapunov functional is a non-negative scalar function. The results obtained modify and extend a number of well-known results. Antonis Parachristodoulou [2] used sum of square decomposition to construct...
Lyapunov function for system with equality, inequality and integral constraints which allows certain non polynomials and non linearity in the vector field to be analysed.

In another paper [5], Gil formulated explicit stability for non-linear retarded system with separate autonomous linear parts in terms of the roots of the characteristic polynomials based on the recent estimates for the matrix resolvent. He discussed the global stability; considered the estimates for the norm of the Green’s function and then derive a bound for a region of attraction of the zero solution.

In this paper, suitable Lyapunov functions will be constructed for a system of linear and non linear second order ordinary differential equations. Basic definitions will also be used to determine the stability in the large of their trivial solutions.

2. Mathematical Formulation

The system of linear second order ordinary differential equation that will be considered in this work is

\[ \ddot{x} + a\dot{x} + bx = 0 \]  \hspace{1cm} (2.1)

where \( a,b, > 0, x \neq 0 \) and the system of non linear second order ordinary differential equation in this work is

\[ \ddot{x} + \dot{f}(x) + h(x) = 0 \]  \hspace{1cm} (2.2)

where \( f(x), h(x) > 0 \) are functions of \( x \) and \( f(0) = h(0) = 0 \).

3. Method of solution

Definition 3.1

Let \( \Omega \in \mathbb{R}^n \) be a neighbourhood of the origin and let \( V : \Omega \rightarrow \mathbb{R} \). Then the function \( V = V(x) \) is said to be:

(i) positive definite if \( V(0) = 0 \) and \( V(x) > 0 \) for \( 0 \neq x \in \Omega \)
(ii) negative definite if \( V(0) = 0 \) and \( V(x) < 0 \) for \( 0 \neq x \in \Omega \)

(iii) positive semidefinite if \( V(0) = 0 \) and \( V(x) \geq 0 \) for \( 0 \neq x \in \Omega \)

(iv) negative semidefinite if \( V(0) = 0 \) and \( V(x) \leq 0 \) for \( 0 \neq x \in \Omega \)

**Definition 3.2**

Consider the continuous system of second order ordinary differential equation

\[
x = f(x) \quad f(0) = 0
\]

(3.2.1)

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be such that solution exists and unique for a given initial data. The trivial solution \( x \equiv 0 \) of (3.2.1) is said to be:

(i) stable (in the sense of Lyapunov) if \( \exists \) a function \( V : \Omega \to \mathbb{R} \) that is positive or negative definite and whose derivative \( \dot{V}(x) = \frac{dV(x)}{dx(t)} = \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} f_i(x) \) with respect to the system (3.2.1) is positive or negative semi definite.

(ii) asymptotically stable if \( \dot{V}(x) \) is negative or positive definite.

4. **Exact solutions**

Starting with the system of linear second order ordinary differential equation (2.1), its equivalent system is

\[
\dot{x} = y, \quad \dot{y} = -ay - bx
\]

(4.1)

We seek a function \( V : \mathbb{R}^2 \to \mathbb{R} \) such that

i) \( V(x, y) \) is positive definite

ii) \( V(x, y) \) is positive or negative semi definite.

Now consider for (2.1), \( 2V(x, y) = k_1x^2 + k_2y^2 + 2k_3xy \)

(4.2)

Differentiating (4.2), we obtain
\[ \dot{V}(x, y) = (k_1 - ak_3 - bk_2)xy + (k_3 - ak_2)y^2 - bk_3x^2 \]  
(4.3)

There are three cases for the stability of trivial solution of (2.1)

**Case 1:** \( \dot{V} \alpha x^2 \)

Here, \( k_1 - ak_3 - bk_2 = 0 \)

\[ k_3 - ak_2 = 0 \quad \text{and} \quad bk_3 > 0 \]

Hence, \( k_3 = ak_2, k_1 = a^2k_2 + bk_2 \) and \( abk_2 > 0 \). Then, fix \( k_2 \equiv 1, k_3 = a, k_1 = a^2 + b \) and \( a, b > 0 \).

Then, (4.2) becomes

\[ 2V_1(x, y) = (a^2 + b)x^2 + y^2 + 2axy = bx^2 + (ax + y)^2 > 0 \]
(4.4)

which is positive definite since \( a, b > 0 \) and \( x, y \neq 0 \). Then, differentiating (4.4)

\[ \dot{V}_1(x, y) = -abx^2 < 0 \]
(4.5)

which is negative semi definite since \( a, b > 0 \) and \( \dot{V}_1(0, \varepsilon) = 0 \)

**Case 2:** \( \dot{V} \alpha y^2 \)

Here \( k_1 - ak_3 - bk_2 = 0, \quad bk_3 = 0 \) and \( k_3 - ak_2 < 0 \)

Assume \( b \neq 0 \), then \( k_3 = 0 \) so that \( k_1 = bk_2 \). Then, fix \( k_2 \equiv 1, k_1 = b \) and using (4.2)

\[ 2V_2(x, y) = bx^2 + y^2 > 0 \]
(4.6)

which is positive definite since \( b > 0 \) and \( x, y \neq 0 \). Then differentiating (4.6),

\[ \dot{V}_2(x, y) = -ay^2 < 0 \]
(4.7)

which is negative semi definite since \( a > 0 \) and \( \dot{V}_2(\varepsilon, 0) = 0 \)

**Case 3:** \( \dot{V} \alpha (x^2 + y^2) \)

To conclude the stability of the trivial solution of (2.1),

\[ 2V_3(x, y) = 2V_1(x, y) + 2V_2(x, y) \]
\[ 2V_3(x, y) = 2bx^2 + (ax + y)^2 + y^2 > 0 \]  
(4.8)

which is positive definite since \( a, b > 0 \) and \( x, y \neq 0 \).

Hence, (4.8) is the suitable Lyapunov function for the system of linear ordinary differential equation (2.1). Differentiating (4.8),

\[ \dot{V}_3(x, y) = -a(bx^2 + y^2) < 0 \]  
(4.9)

which is negative definite since \( a, b > 0 \) and \( x, y \neq 0 \)

Then, extension to the non-linear second order ordinary differential equation (2.2), the equivalent system is

\[ \dot{x} = y, \quad \dot{y} = -yx f(x) - h(x) \]  
(4.10)

From equations (2.1) and (2.2), it is observed that \( a = f(x) \) and \( bx = h(x) \). Then, using the three cases of the linear system above, the three cases for the non-linear system are given below:

**Case 1**  \[ \dot{V} \propto x^2 \]

\[ 2V_1^*(x, y) = 2 \int_0^s h(s) ds + \left[ x f'(x) + y \right]^2 > 0 \]  
(4.4*)

which is positive definite since \( x, y \neq 0 \) and \( xf(x), xh(x) > 0 \)

Differentiating (4.4*), then

\[ 2V_1^*(x, y) = 2xh(x) + 2xf'(x) + yf(x) - yf(x) - h(x) \]
\[ \dot{V}_1^*(x, y) = -x f(x) \left[ x f'(x) + h(x) \right] \]  
(4.5*)

which is negative semi definite since \( \dot{V}_1^*(0, \varepsilon) = 0 \).

**Case 2**  \[ \dot{V} \propto y^2 \]

\[ 2V_2^*(x, y) = 2 \int_0^s h(s) ds + y^2 > 0 \]  
(4.6*)

which is positive definite since \( x, y \neq 0 \) and \( xh(x) > 0 \)

Differentiating (4.6*), then
\[ 2\dot{V}_2^*(x, y) = 2xh(x) + 2y \cdot y \]
\[ \dot{V}_2^*(x, y) = -y^2 f(x) \quad (4.7*) \]

which is negative semi definite since \[ \dot{V}_2^*(\varepsilon, 0) = 0 \]

**Case 3**  
\[ \dot{V} \alpha(x^2 + y^2) \]

To conclude the stability of the trivial solution for (2.2),

\[ 2V_3^*(x, y) = 2V_1^*(x, y) + 2V_2^*(x, y) \]
\[ 2V_3^*(x, y) = 4\int_0^s h(s)ds + [xf(x) + y]^2 + y^2 > 0 \quad (4.8*) \]

which is positive definite since \( x, y \neq 0 \) and \( xf(x), xh(x) > 0 \)

Equation (4.8*) is the Lyapunov function for the non linear system.

Then, differentiating (4.8*),

\[ 2\dot{V}_3^*(x, y) = 2\dot{V}_1^*(x, y) + 2\dot{V}_2^*(x, y) \]

Hence, \( \dot{V}_3^*(x, y) = -f(x)\{xf(x) + xh(x) + y^2}\} \quad (4.9*) \]

which is negative semi definite since \( f(0) = h(0) = 0 \) and \( \dot{V}_3^*(0, \varepsilon) = 0 \)

5. **Analysis of results**

From the results obtained for the three cases of the system of linear second order differential equation, the Lyapunov function constructed in case 3 is positive definite and its derivative is negative definite. Hence, the trivial solution of the linear system is asymptotically stable. On the other hand, the results obtained from three cases of the system of non linear second order differential equation, the Lyapunov function constructed in case 3 is positive definite and its derivative is negative semi definite. Hence, the trivial solution of the non linear system is stable (in the sense of Lyapunov).
References


