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ON THE BI-QUADRATIC DIOPHANTINE EQUATION WITH THREE UNKNOWNS $7\left(x^2-y^2\right)+x+y=8z^4$

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ABSTRACT

We obtain infinitely many non-zero integer triples (x, y, z) satisfying the non-homogenous bi-quadratic equation with three unknowns. Various interesting properties among the values of x, y, z are presented. Some relations between the solutions and special numbers are exhibited.

KEY WORDS: Integral solutions, Bi-quadratic equation with three unknowns, Special numbers.

MSC Classification:11D25

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INTRODUCTION

There is a great interest for mathematicians since ambiguity in homogeneous and non-homogeneous Bi-Quadratic Diophantine Equations [1-4]. In this context, one may refer [5-14] for varieties of problems on the bi-quadratic Diophantine equations with three or four variables. In this paper, bi-quadratic equation with three variables given by $7\left(x^2-y^2\right)+x+y=8z^4$ is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special numbers are exhibited.

METHOD OF ANALYSIS

The non-homogeneous bi-quadratic Diophantine equation to be solved for its distinct non-zero integral solution is

$$7\left(x^2-y^2\right)+x+y=8z^4 \tag{1}$$

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Pattern I:

Introduction of the transformations

$$x=4u^{2}+v, y=4u^{2}-v, z=u, u\neq v\neq 0 \tag{2}$$

in (1) leads to

$$u^{2}=14v+1 \tag{3}$$

whose fundamental solution is

$$u_{0}=13, v_{0}=12 \tag{4}$$

Assume $u_{1}=h-u_{0}, v_{1}=v_{0}+h$ (5)

Substituting (5) in (3), we get

$$h=14+2u_{0} \tag{6}$$

and thus

$$u_{1}=u_{0}+14, v_{1}=2u_{0}+v_{0}+14 \tag{7}$$

Repeating the above process, we obtain

$$\begin{aligned} u_{n}&=u_{0}+14n\\ v_{n}&=2nu_{0}+v_{0}+14*n^{2} \end{aligned} \tag{8}$$

In view of (2), the corresponding non-zero distinct integral solutions to (1) are given by

$$\begin{aligned} x_{n}&=4u_{0}^{2}+114nu_{0}+v_{0}+798n^{2}\\ y_{n}&=4u_{0}^{2}+110nu_{0}-v_{0}+770n^{2}\\ z_{n}&=u_{0}+14n \end{aligned}$$

Pattern II:

Introducing the transformations

$$x=4u^{2}+v^{2}, y=4u^{2}-v^{2}, z=u, u\neq v\neq 0 \tag{9}$$

in (1), it leads to

$$u^{2}=14v^{2}+1 \tag{10}$$

whose general solution is found to be

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$$u_n = \frac{f_n}{2}$$
$$v_n = \frac{g_n}{2\sqrt{14}} \qquad (11)$$

where $f_n = (15+4\sqrt{14})^{n+1} + (15-4\sqrt{14})^{n+1}$
$g_n = (15+4\sqrt{14})^{n+1} - (15-4\sqrt{14})^{n+1}$, $n = -1,0,1,2,...$

Substituting (11) in (9), the corresponding non-zero distinct integer solutions to (1) are obtained as

$$x_n = f_n^2 + \frac{g_n^2}{56}$$
$$y_n = f_n^2 - \frac{g_n^2}{56} \qquad (12)$$
$$z_n = \frac{f_n}{2}$$

The recurrence relations satisfied by x and y are given by

$$x_{n+2} - 898x_{n+1} + x_n = -1760$$
$$y_{n+2} - 898y_{n+1} + y_n = -1824, \quad n = -1,0,1,2....$$

A few numerical examples are given below:

Table 1: Examples

n	x_n	y_n	z_n
-1	4	4	1
0	916	884	15
1	820804	792004	449
2	737079316	711216884	13455

Properties:

- $57y_n = 55x_n + 8$
- $28(x_n - y_n)$ is a perfect square.
- $z_n^2 = 7Y_n + 1$, where $Y_n = (x_n - y_n)$
- Each of the following expressions represents a nasty number:
  - $3(x_n + y_n)$

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- $84\left(x_n^2 - y_n^2\right)$
- $\frac{2}{19}\left(56x_{2n+1} + 4\right)$

❖ Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table below.

Table 2: Parabola

Parabola	$\left(X_{n,} Y_n\right)$
$14X_n^2 = 51300Y_n - 1169600$	$\left(449x_n - x_{n+1} - 880, 56x_{2n+1} + 4\right)$
$14X_n^2 = 49500Y_n - 10890000$	$\left(449y_n - y_{n+1} - 912, 56y_{2n+1} - 4\right)$
$14X_n^2 = 4.13685252 * 10^{10} Y_n - 9.432023746 * 10^{12}$	$\left(403201x_n - x_{n+2} - 792000, 56x_{2n+1} + 4\right)$

❖ Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table below.

Table 3: Hyperbola

Hyperbola	$\left(X_{n,} Y_n\right)$
$900Y_n^2 - 14X_n^2 = 11696400$	$\left(449x_n - x_{n+1} - 880, 56x_n - 110\right)$
$725763600Y_n^2 - 14X_n^2 = 9.432023746 * 10^{12}$	$\left(403201x_n - x_{n+2} - 792000, 56x_n - 110\right)$
$900Y_n^2 - 14X_n^2 = 10890000$	$\left(449y_n - y_{n+1} - 912, 56y_n - 114\right)$

Pattern III:

By introducing the linear transformation

$$x = y + t \tag{13}$$

in (1), it results in

$$7t^2 + t(14y + 1) + \left(2y - 8z^4\right) = 0$$

Which is a quadratic in ‘ t ‘ and solving for t,

$$t = \frac{1}{14}\left\{-(14y + 1) \pm \sqrt{(14y - 1)^2 + 224z^4}\right\} \tag{14}$$

Let $\alpha^2 = 224z^4 + (14y - 1)^2$

and it is satisfied by

$$\begin{aligned} 14y-1 &= 224r^2-s^2 \\ \alpha &= 224r^2+s^2 \\ z^2 &= 2rs \end{aligned} \tag{15}$$

As the values of y and z are to be in integers, choosing $r=2^{2\beta-1}*(14u+1)$, it is seen that

$$\begin{aligned} y &= (14u^2+2u)(56*2^{4\beta}-1)+2^{4\beta+2} \\ z &= 2^{\beta}(14u+1) \end{aligned} \tag{16}$$

In view of (13), we obtain

$$x=(14u^2+2u)(56*2^{4\beta}+1)+2^{4\beta+2} \tag{17}$$

Therefore , (16) and (17) forms the corresponding non-zero distinct integer solutions to (1).

Properties:

- $x(u,\beta)-y(u,\beta)-t_{58,u}\equiv 0(\mathrm{mod}\,31)$, $t_{m,n}$ - Polygonal number of rank n with size m.
- $x(u,\beta)+y(u,\beta)-2^{8\beta+9}t_{100,u}\equiv 0(\mathrm{mod}\ 5)$
- $6\{z(2+14*2^{\beta},\beta)-(2^{\beta}-1)\}$ is a nasty number.

Conclusion:

In this paper, we have made an attempt to find different patterns of non-zero distinct integer solutions to the bi-quadratic equation with three unknowns given by $7(x^2-y^2)+x+y=8z^4$. As bi-quadratic equations are rich in variety, one may search for integer solutions to other choices of bi-quadratic equations with multivariates along with suitable properties.

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