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Stc-M-injective and Stc-self-injective Modules

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ABSTRACT

In this paper we introduce the concepts Stc-M-injective and Stc-self-injective modules which are proper generalizations of M-injective and self-injective respectively. Numerous properties of these generalizations are given. Moreover, we discussed the relationship between Stc-injectivity and Rc-injectivity, also the relation with C-injectivity.

Key words:*St-closed submodules*, *rationally closed submodules, closed submodules, Stc-M-injective modules, M-rc-injective modules, c-self-injective, quasi-injective modules.*

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1. INTRODUCTION

Throughout this paper, we assume every ring R to be an associative ring with identity and every R-modules are unitary right modules.

Let M be an R-module, a module A is called M-injective if for every submodule T of M, any R-homomorphism from T to A can be extended to an R-homomorphism from M to A. A module A is called injective if it is M-injective for every R-module$M$. An R-module A is called self(quasi)-injective, if it is A-injective[9].

A submodule T of an R-module M is called rational in M(briefly $T \leq_r M$), if for every $x, y \in M$ with $x \neq 0$ there exist $r \in R$ such that $yr \in T$ and $xr \neq 0$ [8]. It is well known that a nonzero submodule$T$ of M is called essential (briefly $T \leq_e M$), if $T \cap K \neq (0)$ for each nonzero submodule$K$ of M [9], and a nonzero submodule$T$ of M is called semi-essential (briefly $T \leq_{sem} M$), if $T \cap P \neq (0)$ for each nonzero prime R-submodule P of M [3]. Equivalently, a submodule T of an R-module M is called semi-essential if whenever$T \cap P = (0)$, then $P = (0)$ for every prime submodule P of M [10], where a submodule P of M is called prime, if whenever $rm \in P$ for $r \in R$ and $m \in M$, then either $m \in P$ or $r \in (P_R: M)$ [13].

An R-module M is called fully prime, if every proper submodule of M is a prime submodule [4] .So we have the following implications:

Rational submodules$\Longrightarrow$ Essential submodules $\Longrightarrow$ Semi-essential submodules.

In general, neither of the converses of implication is holds.

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An R-module M is called monoform is every non-zero submodule of M is rational submodule of M[1]. An R-module M is called uniform(semi-uniform) is every non-zero submodule of M is essential(semi-essential) submodule of M[8][2]. So we have the following implications:
Monoform module⟹ Uniform module ⟹ Semi-uniform module .

A submodule H of an R-module M is called rationally closed in M (briefly $H \leq_{rc} M$), if$H$ has no proper rational extension in M,i.e if $H \leq_r K \leq M$then $H = K$ [1].A submodule$H$ of M is called closed submodule (briefly $H \leq_c M$), if H has no proper essential extensions in M, i.e if $H \leq_e K \leq M$ then $H = K$ [6]. A submodule H of an R-module M is called St-closed if H has no proper semi-essential extensions in M, i.e if $H \leq_{sem} K \leq M$ then $H = K$[11] **.**

Let M be R-module. In [5] an R-module A is called M-c-injective, if every R-homomorphism $\alpha: H \to A$, where H is a closed submodule of M, can be extended to an R-homomorphism $\beta: M \to A$. An R-module A is said to be self-c-injective if $A$is A-c-injective.
An R-module M is called C-quasi-injective if, any R-homomorphism $\varphi: H \to M$, where H is a closed submodule of M, can be extended to some $\alpha \in End(M)$ [14]. Obviously that, the concepts of self-c-injective and C-quasi-injective R-modules are the same.
In [2], Abbas, M. S., and Mahdi, S. N. introduced the concepts M-rc-injective and rc-quasi-injective modules. Let M_1 and M_2 be R-modules. Then M_2 is called M_1-rc-injective if every R-homomorphism $f: H \to M_2$, where H is rationally closed submodule of M_1, can be extended to an R-homomorphism $g: M_1 \to M_2$. An R-module M is called rc -injective, if M is N-rc-injective, for every R-module N. An R-module M is called rc –quasi-injective or self-rc –injective, if M is M-rc-injective[2].

Thus we have the following implications:
Injective module⟹quasi-injective module⟹rc-quasi-injective module ⟹c-quasi-injective module.
In general, neither of the converses of implication is hold[2].
In this work, we introduce and study a proper generalization of M-injectivity and Self-injectivity, namely Stc- M-injective and Stc-self-injective modules respectively. Also, we will shows that the class of Stc-injectivity is a weak than the class of C-injectivity.

2. SOMERESULTSON St-CLOSEDSUBMODULES

In this section we introduce some results on St-closed submodule(briefly Stc-submodule) which needed in our work.
Definition 2.1: *An R-module M is called STC-module if every submodule of M is Stc-submodule.*

Remarks and Examples 2.2:
(1) $M = Z_6$as Z-module is STC-module.

(2) $M = Z$as Z-module is not STC-module , since a submodule$T = 5Z$ is not Stc-submodule of Z since$T = 5Z$ is semi-essential submodule of Z (in fact Z is uniform (and hence Z is semi-uniform).

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(3) Every field is not STC-module, since every filed has only one prime submodule which is $< 0 >$ implies $< 0 >$ is semi-essential submodule . Therefor, $< 0 >$ is not Stc-submodule. For example, the Z-modules: Z_2 , Z_3,…Z_p (where p is prime number) are not STC-modules.

(4) Follows [11], if D is a direct summand of an R-module M then not necessary D is Stc-submodule in M, for example: Consider the Z-module,

$Z_{36} = 9Z \oplus 4Z$,it is clear that $9Z$ is a direct summand of Z_{36} but not Stc- submodule in Z_{36}

(5) Every St-closed submodule is rationally closed submodule.

Proof: Let T be an Stc-submodule of an R-module M. Then by [11,Remark(1.3)], T is closed submodule in M. Hence by [1], T is rationally closed submodule. □

(6) the converse of (5), may not be true in general. For example a submodule$2Z$ of Z-module Z_4 is rationally closed but not Stc-submodule, since $2Z$ is semi-essential submodule in Z_4 .

The following implication explain the relation between these submodules
St-closed submodules $\Longrightarrow$ ***Closed submodules*** $\Longrightarrow$ ***Rational closed submodules.***
The converses of this implication may not be true in general.

Proposition2.3: *Every submodule in STC-module is direct summand.*
Proof: Suppose that M be STC-module and H be any a submodule of M. Then, by [definition (2.1)], H is Stc-submodule of M and hence by[11,Reamark(1.3)], H is closed submodule of M. This implies that, every submodule of M is closed submodule, thus by [9, P.139] , H is a direct summand in M, that means, every submodule of M is direct summand. □

An R-module M is extending if and only if every closed submodule of M is direct summand[6].

Corollary 2.4: *Every STC-module is semisimple R-module(and hence extending module).*
The converse of Corollary (2.4) may not be true in general. For example, consider $M = Z_2$ as Z-module. It is well known that, M is semi simple module(and hence extending module), but M is not STC-module, since $< 0 >$ is not Stc-submodule of Z_2.
An R-module M is called fully prime if every submodule of M is prime[4].

Theorem 2.5:*For fully prime R-module M. The following statements are equivalent:*
(i) $M$is STC-module.
(ii) $M$is semi simple module.
Proof: (i) $\Longrightarrow$ (ii) It is follows corollary (2.4).
(ii) $\Longrightarrow$ (i) Suppose that, M is semi simple module and T be any submodule of M **.** Then T is direct summand of M. It is well known that every direct summand is closed submodule, thus T is closed submodule in M. Since M is fully prime R-module then by [11,Remark (1.8)], T is Stc-submodule in M. Therefore, M is STC-module. □

Recall that a singular submodule defined by $Z(M) = \{m \in M: ann(m) \leq_e R\}$. M is called the singular module, if$Z(M) = M$,and M is called a nonsingular module, if $Z(M) = 0$ [8].

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In the following results we explain when the converses of the implication of remark ((2.2)(6)) is true**.**

Firstly, we need the following lemmaswhich appeared in [11]and [1] respectively.

Lemma2.6: *If an R-module M is fully prime, then every nonzero closed submodule in M is an St-closed submodule.*

Lemma2.7:*If an R-module M is non-singular, then every nonzero rationally closed submodule in M is an closed submodule.*

Proposition 2.8: *Let Mbe non-singular fully prime R-module, and T be non-zero submoduleof M. Then the following statements are equivalent:*

(i) T is St-closed submodule

(ii) T is closed submodule

*(iii) T is rationally closed submodule***.**

Proof: $(i) \Leftrightarrow (ii)$follows [11, the remarks (1.3)] and [Lemma (2.6)]

$(ii) \Leftrightarrow (iii)$follows [1, definition (1.5)] and [Lemma (2.7)]

$(i) \Leftarrow (iii)$Suppose that T is rationally closed submodulein M. Since M is non-singular then by [Lemma (2.7)], T is closed submodule. So that, by [Lemma (2.6)], T is St-closed submodule. □

3. Stc-M-INJECTIVE and Stc-self-INJECTVE MODULES

Definition 3.1: *Let M and A be R-modules. Then A is called $Stc - M$-injective if every R-homomorphism $\varphi: T \to A$, where T is St-closed submodule of M, can be extended to an R-homomorphism $\theta: M \to A$. An R-module A is called Stc-injective, if A is $Stc - M$-injective, for every R-module M. An R-module A is called Stc–quasi-injective or Stc-self-injective, if A is $Stc - A$-injective.*

Remark and Examples 3.2:

(1) It is easy to show that, if A injective module then A is also Stc-injective module.

(2) The converse of (1) is not true in general, as following example:

Let $A = Z_2$ as Z-module. Since A is the only St-closed submodule of A , hence A is Stc-injective module. But, on other hand A=Z_2 is not divisible Z-module , 0=2$Z_2 \neq Z_2$. Therefore, Z_2 is not injective.

3) Every semi-uniform (and hence every uniform) R-module is Stc-self-injective module.

Proof:- let A is semi-uniform R-module, then every submodule of A is semi-essential submodule. So, that A is the only St-closed submodule in A. Therefore, A is Stc-self-injective. □

(4) Every self-injective R-module is Stc-self-injective module.

(5) The converse of (4) is not true in general. Consider the module Z as Z-module. It is well known that, Z is not quasi-injective. But, since Z is semi-uniform(in fact Z is uniform), hence by (3), Z is Stc-self- injective. This show that, Stc-self-injective is proper generalization of quasi-injective.

(6) It is obvious that if A_2 isA_1-c-injective (and henceA_1-c-injective) then A_2 is also Stc-A_1-injective (this means that, Stc-injectivity is weak than c-injectivity). Thus we get the following implication:

Injective ⇒Quasi-injective ⇒RC-quasi-injective⇒C-quasi-injective⇒Stc-quasi-injective

In the following proposition we will shows when the concepts Rc-injectivity, c-injectivity and Stc-injectivity are equivalents.

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Proposition 3.3: *Let Mbe non-singular fully prime R-module and A be any R-module. Then the following statements are equivalent:*
(i) A is $rc-M$-injective
(ii) A is $c-M$-injective
(iii) A is $Stc-M$-injective.
Proof: $(i) \Rightarrow (ii) \Rightarrow (iii)$It is clear.
$(iii) \Rightarrow (i)$ Suppose that, $A$is $Stc-M$-injective**.** Let K be rationally closed submodule of M with $\theta: K \to A$ be a homomorphism. Since Mbe non-singular fully prime R-module, then by [proposition (2.8)], K is St-closed submoduleof M. So that, by $Stc-M$-injectivity of A, there is an R-homomorphism $f: M \to A$ such that extend θ. This means, $A$is $rc-M$-injective. □

Corollary 3.4: *Let Abe non-singular fully prime R-module. Then the following statements are equivalent:*
(i) A is $rc-Self$-injective
(ii) A is $c-Self$-injective
(iii) A is $Stc-Self$-injective.

In the following results we introduce some general properties of $Stc-M$ −injectivity and $Stc-self$- injectivity.

Proposition 3.5:*Let K ,T be two isomorphic R-modules and M be any R-module. If K is $Stc-M$ -injective then T is $Stc-M$- injective.*
Proof: The proof is routine.□

Proposition 3.6: *Let $M$and $A_i (i \in I)$ be R-modules. Then $A = \prod_{i \in I} A_i$ is $Stc-M$-injective if and only if A_i is $Stc-M$-injective, for every $i \in I$.*
Proof: First direction: Assume that, $\prod_{i \in I} A_i$ is $Stc-M$-injective. Let $\tau: T \to A_i$ be an R-homomorphism(where T is St-closed submodule of M). Now, consider the following mapping $\pi_i: \prod_{i \in I} A_i \to A_i$ and $\sigma_i: A_i \to \prod_{i \in I} A_i$ (where, π_i is the natural projection of A intoA_i and σ_i is the natural injection fromA_i into A, for every $i \in I$.). Since $\sigma_i \circ \tau$ is an R-homomorphism from T into A and we have A is $Stc-M$-injective then there exists an R-homomorphism $\theta: M \to A$ such that $\sigma_i \circ \tau = \theta \circ \mu$(where$\mu$ is inclusion map from T into M). Claim that, there exists an R-homomorphism $f: M \to A_i$ such that $= f \circ \mu$, to show this, for every $t \in T$, $\tau(x) = I_{A_i} \circ f(x) = \big((\pi_i \circ \sigma_i) \circ \tau\big)(x) = (\pi_i \circ \big((\sigma_i \circ \tau)(x)\big) = (\pi_i \circ \big((\theta \circ \mu)(x)\big) = \big((\pi_i \circ \sigma_i) \circ \mu\big)(x) = (f \circ \mu)(x)$ (where I_{A_i} is identity mapping of A_i for every $i \in I$). Therefore, this show that, A_i is $Stc-M$-injective, for every $i \in I$.
Second direction: Assume that, A_i is $Stc-M$-injective, for every $i \in I$. Let $g: T \to A = \prod_{i \in I} A_i$ be an R-homomorphism(where T is St-closed submodule of M).Then, by assumption and for every homomorphism $\pi_i \circ g$ from T into A_i , there exists an R-homomorphism $\beta_i: M \to A_i$ such that $\pi_i \circ g = \beta_i \circ \mu$ thus by[9,Theorem (4.1.6)] there exists an R-homomorphism $\beta: M \to A$ with $\pi_i \circ \beta = \beta_i$. It is not hard to shoe that $g = \beta \circ \mu$. Therefore, $\prod_{i \in I} A_i$ is $Stc-M$-injective. □
As an immediate consequence of propositions (3.6) there is the following corollary.

Corollary3.7: *Let$A = T \oplus T'$. If A is $Stc-M$ −injective module, then T is $Stc-M$ −injective.*
□

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Recall that an R-module M is called chained, if for each sub modules A and B of M either $A \leq B$ or $B \leq A$ [12].

Lemma3.8: [11]
(i) *For a chained R-module* M. *If* $A \leq Stc\ B$ *and* $B \leq Stc\ M$, *then* $A \leq Stc\ M$.
(ii)The intersection of two Stc-submodules is also Stc-submodule.
(iii)For a fully prime R-module M. *If* $(0) \neq A \leq Stc\ B$ *and* $B \leq Stc\ M$, *then* $A \leq Stc\ M$.
Now, we are ready to consider the following results.

Proposition 3.9: *Let* M *be a chained* R*-module and* W *be any* R*-module with* W *is* $Stc - M$*-injective. Then the following statements hold.*

(i) *If* $T \leq_{Stc} M$ *then* W *is* $Stc - T$*- injective.*
(ii) *If* $T \leq_{Stc} M$ *and* $K \leq_{Stc} M$ *then* W *is* $Stc - T \cap K$*- injective.*

Proof:((i) Let $T \leq_{Stc} M$, $L \leq_{Stc} T$ and $\varphi: L \to W$ be R-homomorphism. By [Lemma(3.8)(i)] we have $L \leq_{Stc} M$, hence by $Stc - M$- injectivity of W, there exists a R-homomorphism $f: M \to W$ such that $f i_T i_L = \varphi$ where $i_L: L \to T$ and $i_T: T \to M_1$are inclusion maps.Let$\beta = g \circ i_H$. Clearly,β is R-homomorphism, and $\beta = f\ i_L = f i_T i_L = \varphi$. Therefore, W is $Stc - T$ -injective. □
(ii)It is clearly, follows from statement (i) and [Lemma(3.8)(ii)]. □

Corollary 3.10:*Let* A_1 *and* A_2 *be any* R*-modules. If* $A_1 \oplus A_2$ *is* $Stc - self -$ *injective, then* A_1 *and* A_2 *are both* $Stc - self -$*injectives* . □

Corollary 3.11: *An* R*-module* M*is* $Stc - self -$*injective if* M *is* $Stc - B -$*injective for every Stc-submodule*B *of* M.
Proof: It follows from proposition (3.9). □

Proposition 3.12: *A non- zero direct summands of* $Stc - self -$ *injective fully prime R-module is* $Stc - self -$*injective* .
Proof: Let M be fully prime module with M is $Stc - self -$ injective and $(0) \neq T$ be any direst summand of M . Hence $M = T \oplus T'$ for some submodule T' of M. Let K be an Stc- submodule in T and $f: K \to T$ be any R-homomorphism. Since, every direct summand is closed submodule then T is closed submodule in M, hence by [Lemma (3.8)(iii)],T is an Stc- submodule in M, implies K is an Stc- submodule in M then by $Stc - self -$ infectivity of M , there exists R-homomorphism $\beta: M \to M$ such that $\beta i_T i_k = j_T f$, where $j_T$is injection mapping from T to M and i_k is inclusion from K into T . Let $\rho: M \to T$ be the projection map. Define $g: T \to T$ by $g(t) = \rho(\beta(t))$ for any $t \in T$. Follows that, for each $x \in K$, $g(x) = g i_k(x) = \rho\beta i_K(x) = \rho j_A \circ f(x) = f(x)$. Hence T is$Stc - self -$ injective.□

Proposition3.13:*Let* M *be a chained* R*-module and* W *be any* R*-module.* W*is*$Stc - M$*-injective if and only if every direct summand* K *of* W *is* $Stc - T$*-injective for every Stc-submodule*T*of* M.

Proof: Suppose that, W is $Stc - M$- injective. Let K is a direct summand of W then by [Corollary (3.7)], K is $Stc - M$-injective. Now, let H is Stc-submodule of T then by [Lemma

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(3.8)(i)], H is Stc-submodule of M, hence it is clear that by [Proposition(2.9)], K is $Stc-T$-injective. Conversely, it is directly from first direction, since we have W is direct summand of W and M is Stc-submodule of M. □

In general, not every Stc-submodule is direct summand[11]. For example, the submodule$6Z$ is Stc-submodule of Z_{12} as Z-module, but clear that, $6Z$ is not direct summand of Z_{12}.

In the following results we show that when the Stc-submodule is direct summand

Proposition3.14:*Let A be a $Stc-self$- injective R-module and let T be an Stc-submodule of A . If $T \cong A$ then T is a direct summand of A .*

Proof: In a similar way of [2], we can prove it. □

In the following proposition, we give a characterization of $Stc-M$-injective modules.

Proposition3.15: *Let $M = A_1 \oplus A_2$where A_1 and A_2 be two R-modules.Then the following statements are equivalent:*

(i) A_2 is $Stc-A_1$-injective

(ii) For every (St-closed) submodule$T$ of M such that the intersection of T with A_2 equal zero and $\pi_1(T)$ is St- closed submodule of A_1(where$\pi_1$is the natural projection of M intoA_1),there exists a submodule T' of$M$ such that $T \leq T'$ and $M = A' \oplus A_2$.

Proof: $(i) \Rightarrow (ii)$ Assume that, T be submodule of M such that $T \cap A_2 = 0$ and let A_2 is $Stc-A_1$-injective. Let $\pi_i: M \to A_i$,$(i = 1,2)$ be the projective mapping and $\pi_1(T)$ is St-closed submodule ofA_1. As$H \cap N_2 = 0$, the restriction of π_1 to H is an R-isomorphism between $H$and $\pi_1(T)$. Define $g: \pi_1(T) \to A_2$ such that $g(x) = \pi_2[(\pi_1)^{-1}(x)]$ (where, $\forall x \in \pi_1(T)$), $\exists\, t \in T$ such that $x = \pi_1(t)$) .It is easy to check that g is well-define and R-homomorphism. Since $\pi_1(T) \leq_{Stc} A_1$ and A_2 is $Stc-A_1$-injective, then g can be extended to an R-homomorphism $\theta: A_1 \to A_2$. Define $T' = \{t + \theta(t), t \in A_1\}$. Clearly, T' is a submodule of $M$and $M = T' \oplus A_2$. For every $t \in T$ *we have* $t = a_1 + a_2$ where $a_1 \in A_1$ and $a_2 \in A_2$ .So, since $\pi_1(t) = a_1$ and $\pi_2(t) = a_2$ with $\theta(\pi_1(t)) = g(\pi_1(t)) = \pi_2(t)$ hence, we get $t = \pi_1(t) + \pi_2(t) = \pi_1(t) + \theta(\pi_1(t)) \in T'$. Therefore, T is a submodule of T'.

$(i) \Leftarrow (ii)$, Assume that, the statement (ii) holds. Let $W \leq_{Stc} N_1$ and let $\psi: W \to A_2$ be an R-homomorphism. Define $T = \{w - \psi(w), w \in W\}$. Clearly, T is a sub module of M such that $T \cap A_2 = 0$. It is easily to check that $\pi_1(T) = W$ and so that $\pi_1(T) \leq_{Stc} A_1$. Then by (ii), there exists a submodule$T'$of M such that $T \leq T'$and $M = T' \oplus A_2$. Let $\pi_2: M \to A_2$ denote the projection with ker$\pi_2 = T'$ and let $\phi: A_1 \to A_2$ be the restriction of $\pi$to A_1. For every $w \in W$, $\phi(w) = \pi(w) = \pi[w - \psi(w) + \psi(w)] = \pi(w - \psi(w)) + \pi(\psi(w)) = \psi(w)$ and, therefore, ϕ extends ψ.Thus A_2 is $Stc-A_1$-injective. □

In the following, we characterize injective R-modules in terms of Stc-injectivity.

Proposition3.16:*For an R-module A. $A$is injective if and only if A is $Stc-B$- injective for any R-module B.*

Proof: (i) ⟹ (ii) It is clearly.

(ii) ⟹ (i)As the same way of proof proposition (2.10). □

Recall an R-module M be projective, if for each epimorphism $\theta: B \to A$(A and B be any two R-module) every R-homomorpism $\alpha: M \to A$ there is a homomorphism $\beta: M \to B$ with $\alpha = \theta\beta$ [9, p117].

Theorem 3.17: *Let W be $Stc - B$- injective for any projective R-module .Then the following statements holed.*

(i) $\frac{W}{T}$*is*$Stc - B$*- injective (where T is any submodule of W).*
(ii) *Any Stc-submodule of B is projective*

Proof: (i) $\Longrightarrow$ (ii) Suppose that T is any submodule of W such that $\frac{W}{T}$ is $Stc - B$- injective. Now, Let M be injective R- module and $\frac{M}{F}$ be any factor module of M with an R-epimorphism : $M \to \frac{M}{F}$. Clearly that,$\frac{M}{F}$ is $Stc - B$- injective. Suppose that L is an Stc- submoduleof B, then by B-rc-injectivity of $\frac{M}{F}$, every R-homomorphism $\beta: L \to \frac{M}{F}$ can be extended to an R-homomorphism $\tau: B \to \frac{M}{F}$. Since$B$ is projective, then there exists an R-homomorphism $f: B \to M$ lifts τ .Thus obviously,$f|_L$ lifts β.This implies that L is projective submodule of B.
(ii) $\Longrightarrow$ (i) Let L is Stc- submodule of B, then L is projective. Let $\frac{W}{T}$ and W be two R-modules with an R-homomorphism $\tau: W \to \frac{W}{T}$. Consider $\rho: L \to \frac{W}{T}$ is an R-homomorphism and W is $Stc - B$ -injective. Then by projectivity of L, there exists an R-homomorpism $\theta: L \to W$ lifts ρ . Since W is $Stc - B$ -injective, then there exist an R-homomorphism $\beta: B \to W$extends θ. Thus, clearly $\theta\beta: B \to \frac{W}{T}$ extends ρ.□
If every submodule of M is a projective then the R-module A is called a hereditary module [7.]. Then the following result follows theorem (2.17).

Corollary 2.18:*Let A be a hereditary R-module .Then any factor R-module of an $Stc - A$-injective R-module is $Stc - A -$injective.* □

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