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TRANSITIVITY, RANK, AND SUBORBITSOF THE CYCLIC GROUP C_n ACTING ON UNORDERED SUBSETS

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ABSTRACT
The study aims at determining the rank and subdegrees of the cyclic group, C_n= <(12…n)> acting on $X^{(r)}$, the set of unordered subsets of X={1, 2, …, n}. It has been shown that the action of C_n on $X^{(r)}$is transitive if and only if$r$=1, r=n-1 or$r$=n. The rankfor transitive actions has been shown to be$n$. The number of self paired suborbits has been computed and conditions for paired suborbits discussed. The results have shown that the action of C_n on X is equivalent to that of C_n on$X^{(n-1)}$.
Key Words: Cyclic group, Suborbits, Transitive action, Rank, Unordered set.

INTRODUCTION
The rank and subdegrees of $S_n$acting on various r- element subsets of X={1, 2, …, n} have been studied ([5], [7], [8]). Kamuti et al.,[6] investigated some properties of Γ_∞ (the stabilizer ofinfinity inΓ=PSL(2, $\mathbb{Z}$)) acting on the set of integers. The dihedral group$D_n$acting on ordered and unordered subsets of$X$ has also been consideredwith regard to rank , subdegrees and suborbits[2], [3]. However, the study of the cyclic group has not received much attention. Section 2 outlines some preliminary results which have been used to compute the main results. Section 3 discusses the aspects of transitivity, rank and subdegrees. Properties of suborbits have also been examined in this section. The results have been discussed and concluded in section 4.

Notations and Preliminaries
Notation 2.1
The symbol G denotes the cyclic group C_n=<g=(12…n)>; |G|, the order of a group G; {1, 2, …, r}, an unordered r-element set; G_x={$g\in G$:gx=x}, the stabilizer of x in G, $stab_G(x)$; |$Fix(g)$|={$x\in X$: gx=x}, the number of elements in the fixed point set of g.

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Definition 2.2

Let G act on a set X and $x \in X$. The orbit of x is the set; $orb_G(x)=\{gx \in X \mid g \in G\}$. If the action has only one orbit, then G is said to act transitively on X.

Definition 2.3

Let G be a group acting transitively on a set X. The G_x–orbits on X; $\Delta_0=\{x\}$, Δ_1, …, Δ_{m-1} are known as suborbits of G. The rank of G in this case is m and the cardinalities, $|\Delta_i|$ $(i = 0, 1, \ldots, m\text{-}1)$ are the subdegrees of G. It can be shownthat both m and the cardinalities of the suborbits are independent of the choice of x in X.

Theorem 2.4 (Orbit-Stabilizer Theorem [9]. p. 72)

Let G be a group acting on a finite set X with x in X. The size of the orbit of x in G is the index $|G{:}\, stab_G(x)|$. Thus, $|orb_G(x)| = |G{:}\, stab_G(x)|$.

Theorem 2.5 (Cauchy-Frobenius Lemma [4]. p. 98)

Suppose G is a group acting on a finite set X. The number of G-orbits on X is given by $\frac{1}{|G|}\sum|Fix(\mathrm{g})|$.

Definition 2.6

Let G act transitively on a set X and let Δ be an orbit of G_x on X. Define $\Delta^*=\{gx \mid g \in G, x \in g\Delta\}$. Then Δ^* is also an orbit of G_x and is called the G_x-orbit paired with Δ. If $\Delta=\Delta^*$, then Δ is said to be self-paired.

Theorem 2.7([1])

Let G act transitively on a set X, and suppose $g \in G$. The number of self-paired suborbits of G is given by

$$\frac{1}{|G|}\sum_{\mathrm{g}\in G}|Fix(\mathrm{g}^2)|.$$

Definition 2.8

Let (G_1, S_1) and (G_2, S_2) be permutation groups, where G_i acts on S_i. The permutation isomorphism, $(G_1, S_1) \cong (G_2, S_2)$, means that there exists a group isomorphism $\phi: G_1 \to G_2$ and a bijection $\Theta: S_1 \to S_2$ so that $\Theta(gs)=\phi(g)\Theta(s)$ forall $g \in G_1$, $s \in S_1$.

MAIN RESULTS

3.1 Transitivity, rank and suborbits of $G=C_n$ on $X^{(r)}$

The action of G on $X^{(r)}$ is defined by;

$h\{x_1, x_2, \ldots, x_r\}=\{h(x_1), h(x_2), \ldots, h(x_r)\}$, for every h in G and $\{x_1, x_2, \ldots, x_r\}$ in $X^{(r)}$.

Theorem 3.1.1

The action of G on $X^{(r)}$ is transitive if and only if $r=1$, $r=n\text{-}1$ or $r=n$.

Proof:

Let G act on $X^{(r)}$ and suppose $h \in G$. Then h fixes an element in $X^{(r)}$ if and only if h is the identity. The number of elements in $X^{(r)}$ fixed by h, in this case, is $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. Using Theorem 2.5, the number of G-orbits on $X^{(r)}$ is $\frac{1}{n}\left(\frac{n!}{(n-r)!r!}\right) = \frac{(n-1)!}{(n-r)!r!}$. If the action is transitive, then $\frac{(n-1)!}{(n-r)!r!} = 1$, $\Rightarrow (n\text{-}1)!=(n\text{-}r)!r!$, $\Rightarrow r=1$ or $r=n\text{-}1$. Conversely, if $r=1$ or $r=(n\text{-}1)$, then the number of G-orbits on $X^{(r)}$ is 1 and the action is transitive. Clearly, every $h \in G$ fixes 1 element in $X^{(n)}$. The number of G-orbits on $X^{(n)}$, in this case, is 1 and the action is transitive. □

However, the action of G on $X^{(n)}$ is trivial and the study concentrates on the non-trivial actions.

3.1.2 Rank, subdegrees and suborbits of *G*acting on *X*

Theorem 3.1.2.1

The rank of G on X is n and the length of each suborbit is 1.

Proof:

Let G_1 act on X. From Theorem 3.1.1, $G_1$is the trivial group. The number of G_1-orbits on $X$is then n and the size of each suborbit is 1. Clearly, the subdegrees are; 1, 1, …, 1(n ones). The n suborbits of G on X are as follows; $\Delta_0=\{1\}$, $\Delta_1=\{2\}$, …, $\Delta_{n-1}=\{n\}$, where $\Delta_i=\{i+1\}$, i=0, 1, …, n-1. □

Theorem 3.1.2.2

Let G act on X. If Δ_i and Δ_j areorbits of G_1 on X, then Δ_i and $\Delta_j$are paired if and only if$i+j=0 \ mod \ n$.

Proof:

Suppose Δ_i and $\Delta_j$are paired suborbits of G. Then there existg^kin G, such that $g^k\Delta_0=\Delta_j$and $g^k\Delta_i=\Delta_0$, from Definition 2.6.Now, g^k maps every element t in Δ_i to $t+k \ mod \ n$. It follows, $1+k=j+1$ and $i+1+k=1 mod \ n$, $\Longrightarrow i+j=0 \ mod \ n$. Conversely, if $i+j=0 \ mod \ n$, then, $g^j\Delta_0=\Delta_j$ and $g^j\Delta_i=\Delta_0$. Hence, $\Delta_i$and $\Delta_j$are paired suborbits.

Corollary 3.1.2.3

Let G act on X. Then Δ_i is self- paired if and only if i=0 or $i=n/2 \ mod \ n$.

Proof:

From Theorem 3.1.2.2, Δ_i is self- paired ifand only if$i=j mod \ n$. It follows,i=0 or $i=n/2 \ mod \ n$.

Theorem 3.1.2.4

The number of self- paired suborbits of Gon $X$is 1 when n is odd and 2 when n is even.

Proof:

Let $x\in X$ and $h\in G$. When n is odd, h^2 fixes x if and only if$h$ is the identity. Thus, $\sum|Fix(h^2)| = n$. By Theorem 2.7, the number ofself -paired suborbits is 1.

When n is even, h^2 fixes x if $h$is the identity or h is a rotation of 180^0. Hence, $\sum|Fix(h^2)| = 2n$, and the number of self- paired suborbits is 2.

3.1.3 Rank, suborbits and subdegrees of *G*acting on $X^{(n-1)}$

Theorem 3.1.3.1

The rank of G on $X^{(n-1)}$ is n and the length of each suborbit is 1.

Proof:

Suppose $\{1,2, \ldots, n-1\}\in X^{(n-1)}$ and $G_{\{1, 2, \ldots, n-1\}}$ is the stabilizer of {1, 2, …, n-1} in G. From Theorem 3.1.1, $G_{\{1, 2, \ldots, n-1\}}$ is the trivial subgroup of G. The number of $G_{\{1, 2, \ldots, n-1\}}$-orbits is n and the length of each suborbit is 1. □

The n suborbits of G on $X^{(n-1)}$ are as follows;

$\Delta_0=\{1, 2, \ldots, n-1\}$, $\Delta_1=\{2, 3, \ldots, n-1, n\}$, …, $\Delta_{n-1}=\{n, 1, 2, \ldots, n-2\}$, where $\Delta_i=\{i+1, i+2, \ldots, i-1\}$. Clearly, the subdegrees are; 1, 1, …, 1 (n ones).

Theorem 3.1.3.2

Let G act on $X^{(n-1)}$. Then Δ_i and Δ_j are paired suborbits of Gif and only if $i+j=0 \ mod \ n$.

Proof:

Suppose $\Delta_i$and Δ_j are paired suborbits of G. Then there existg^kin G such that $g^k\Delta_0=\Delta_j$and $g^k\Delta_i=\Delta_0$, by Definition 2.6. It follows, $1+k=j+1$ and $i+1+k=1$, $2+k=j+2$ and $i+2+k=2$, …, $n-1+k=j-1$ and $i-1+k=n-1 \ modn$. Hence, $i+j=0 \ mod \ n$. Conversely, if$i+j=0 \ mod \ n$,then $g^j\Delta_0=\Delta_j$and $g^j\Delta_i=\Delta_0$. It follows,Δ_i and Δ_j are paired.

Corollary 3.1.3.3
Let G act on $X^{(n-1)}$. Then Δ_i is a self- paired suborbit of G if and only if i=0 or $i=n/2$.
Proof:
From Theorem 3.1.3.2, Δ_i is self-paired if and only if $i=j \ mod\ n$. It follows, if i=0 or $i=n/2 \ mod\ n$.
Theorem 3.1.3.4
The number of self- paired suborbits of G acting on $X^{(n-1)}$ is 1 when n is odd and 2 when n is even.
Proof:
Let $A \in X^{(n-1)}$ and $h \in G$. Now, h^2 fixes A if and only if h^2 is the identity. When n is odd, this is possible only if h is the identity. Thus, $\sum |Fix(h^2)| = n$. By Theorem 2.7, the number ofself-paired suborbits is 1.
When n is even, h^2 fixes A if $h$is the identity or h is a rotation of 180^0. It follows, $\sum |Fix(h^2)| = 2n$. The number of self- paired suborbits, in this case, is 2.

RESULTS, DISCUSSION AND CONCLUSION
The action of G on X induces a corresponding action of G on $X^{(n-1)}$. FromTheorem 3.1.1,it has been shown that the stabilizer of a point in each of the two actions is the identity, where$G_x=G_{X/x}$.It has also been revealed that the rank of G is n, in each of the actions, as proved in Theorems 3.1.2.1and 3.1.3.1. Properties of suborbits of G on X correspond to thoseof G on $X^{(n-1)}$, as evidenced in Theorems 3.1.2.2 and 3.1.3.2 respectively. The correspondence is alsoclearfrom Theorems 3.1.2.4 and 3.1.3.4 respectively.The following conclusion suffices.
Theorem 4.1
The action of G on X is equivalent to the action of G on $X^{(n-1)}$.
Proof:
Let (G_1, X) and $(G_2, X^{(n-1)})$ be the action of G on X and the action of G on $X^{(n-1)}$, respectively.Using Definition 2.8, the map $\phi: G_1 \rightarrow G_2$ is such that $\phi(g)=g$, for all $g \in G_1$. Define $\Theta: X \rightarrow X^{(n-1)}$ such that$\Theta(x)=X|x$, for all $x \in X$. Now, $\Theta(gx)=X|gx=g(X|x)=\phi(g)\Theta(x)$. □

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