

Some Fixed Point Results in Complex Valued Metric Spaces with Point-dependent Contractive Conditions

Madhubala Kasar¹, Anil Kumar Dubey²

¹Government Higher Secondary School, Titurdih,

Durg, Chhattisgarh 491001, INDIA

²Department of Mathematics, Bhilai Institute of Technology,

Bhilai House, Durg 491001, INDIA

Abstract: In the present paper, we prove some common fixed point for a pair of mappings satisfying certain point-dependent contractive conditions in complex valued metric spaces. Our results generalize and extend the several known results in the literature.

Key Words: Complex valued metric spaces, Contractive maps, Cauchy Sequence, Common fixed point.

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1. Introduction and Preliminaries

In 2011, Azam et. al. [1] introduced the notion of complex valued metric spaces and proved some fixed point theorems for a pair of mappings for rational type contractive condition. This idea is intended to define rational expressions which are not meaningful in cone metric space and thus many such results of analysis cannot be generalized to cone metric spaces but to complex valued metric spaces.

Complex valued metric space is useful in many branches of Mathematics, including algebraic geometry, number theory, applied Mathematics as well as in

physics, mechanical engineering, thermodynamics and electrical engineering. After establishment of complex valued metric spaces, many authors have contributed with different concepts in this space. One can see in [2 and 4-12].

In this paper, we prove and establish some common fixed point results for a pair of mappings satisfying rational expressions having point-dependent control functions as coefficients in complex valued metric spaces. Our results generalize and extend the results of Azam et. al.[1], Dubey et. al.[3], Rouzkard et. al.[4], Sitthikul et. al.[6], N. Singh et. al.[8] and Sintunavarat et. al.[12].

Consistent with Azam et. al.[1], the following definitions and results will be needed in the sequel.

Let \mathbb{C} be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order \preceq on \mathbb{C} as follows:

$z_1 \preceq z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. It follows that $z_1 \preceq z_2$ if one of the following conditions is satisfied:

- (1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
- (2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
- (3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
- (4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we will write $z_1 \prec z_2$ if $z_1 \neq z_2$ and one of (2), (3) and (4) is satisfied, also we will write $z_1 < z_2$ if only (4) is satisfied.

Definition 1.1[1]. Let X be a non-empty set. A mapping $d: X \times X \rightarrow \mathbb{C}$ is called a complex valued metric on X if the following conditions are satisfied:

- (1) $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

In this case, we say that (X, d) is a complex valued metric space.

Example 1.2. Let $X = \mathbb{C}$ be a set of complex number. Define $d: X \times X \rightarrow \mathbb{C}$ by

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|,$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is a complex valued metric space.

Definition 1.3[1]. Suppose that (X, d) is a complex valued metric space.

- (1) We say that a sequence $\{x_n\}$ is a Cauchy sequence if for every $0 < c \in \mathbb{C}$ there exists an integer N such that $d(x_n, x_m) < c$ for all $n, m \geq N$.
- (2) We say that $\{x_n\}$ converges to an element $x \in X$ if for every $0 < c \in \mathbb{C}$ there exists an integer N such that $d(x_n, x) < c$ for all $n \geq N$. In this case, we write $x_n \xrightarrow{d} x$.
- (3) We say that (X, d) is complete if every Cauchy sequence in X converges to a point in X .

Lemma 1.4 [1]. Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 1.5 [1]. Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is Cauchy sequence if and only if $|d(x_n, x_{n+m})| \rightarrow 0$ as $n \rightarrow \infty$.

2. Main Result

We start to this section with the following observation.

Proposition 2.1[8]. Let (X, d) be a complex valued metric space and $S, T: X \rightarrow X$. Let $x_0 \in X$ and defined the sequence $\{x_n\}$ by

$$\begin{aligned} x_{2n+1} &= Sx_{2n}, \\ x_{2n+2} &= Tx_{2n+1}, \quad \forall n = 0, 1, 2, \dots \end{aligned} \tag{2.1}$$

Assume that there exists a mapping $\succ: X \times X \times X \rightarrow [0, 1)$ such that

$\succ(TSx, y, a) \leq \succ(x, y, a)$ and $\succ(x, STy, a) \leq \succ(x, y, a)$ for all $x, y \in X$, and for a fixed element $a \in X$ and $n = 0, 1, 2, \dots$. Then

$$\succ(x_{2n}, y, a) \leq \succ(x_0, y, a) \text{ and } \succ(x, x_{2n+1}, a) \leq \succ(x, x_1, a).$$

Lemma 2.2 [6]. Let $\{x_n\}$ be a sequence in X and $h \in [0,1)$. If $a_n = |d(x_n, x_{n+1})|$ satisfies $a_n \leq ha_{n-1}, \forall n \in N$, then $\{x_n\}$ is a Cauchy sequence.

Our main theorem runs as follows:

Theorem 2.3. Let (X, d) be a complete complex valued metric space and $S, T: X \rightarrow X$ be mappings. If there exist mappings $\alpha, \beta, \gamma: X \times X \times X \rightarrow [0,1)$ such that for all $x, y \in X$:

$$\begin{aligned} \text{a) } & \alpha(TSx, y, a) \leq \alpha(x, y, a) \text{ and } \alpha(x, STy, a) \leq \alpha(x, y, a), \\ & \beta(TSx, y, a) \leq \beta(x, y, a) \text{ and } \beta(x, STy, a) \leq \beta(x, y, a), \\ & \gamma(TSx, y, a) \leq \gamma(x, y, a) \text{ and } \gamma(x, STy, a) \leq \gamma(x, y, a), \\ \text{b) } & d(Sx, Ty) \lesssim \alpha(x, y, a)d(x, Sx) + \beta(x, y, a)d(y, Ty) \\ & \quad + \gamma(x, y, a)d(x, y), \end{aligned} \tag{2.2}$$

$$\text{c) } \alpha(x, y, a) + \beta(x, y, a) + \gamma(x, y, a) < 1, \tag{2.3}$$

then S and T have a unique common fixed point.

Proof: Let $x, y \in X$, from (2.2), we have

$$\begin{aligned} d(Sx, TSx) & \lesssim \alpha(x, Sx, a)d(x, Sx) + \beta(x, Sx, a)d(Sx, TSx) \\ & \quad + \gamma(x, Sx, a)d(x, Sx), \end{aligned}$$

so that

$$\begin{aligned} |d(Sx, TSx)| & \lesssim \alpha(x, Sx, a)|d(x, Sx)| + \beta(x, Sx, a)|d(Sx, TSx)| \\ & \quad + \gamma(x, Sx, a)|d(x, Sx)|. \end{aligned} \tag{2.4}$$

Similarly, from (2.2) we have

$$\begin{aligned} d(STy, Ty) & \lesssim \alpha(Ty, y, a)d(Ty, STy) + \beta(Ty, y, a)d(y, Ty) \\ & \quad + \gamma(Ty, y, a)d(Ty, y) \end{aligned}$$

so that

$$|d(STy, Ty)| \leq \alpha(Ty, y, a)|d(Ty, STy)| + \beta(Ty, y, a)|d(y, Ty)| + \gamma(Ty, y, a)|d(Ty, y)|. \quad (2.5)$$

Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined by (2.1). We show that $\{x_n\}$ is a Cauchy sequence. From Proposition 2.1 and inequalities (2.4), (2.5) and for all $k = 0, 1, 2, \dots$, we obtain

$$\begin{aligned} |d(x_{2k+1}, x_{2k})| &= |d(STx_{2k-1}, Tx_{2k-1})| \\ &\leq \alpha(Tx_{2k-1}, x_{2k-1}, a)|d(Tx_{2k-1}, STx_{2k-1})| \\ &\quad + \beta(Tx_{2k-1}, x_{2k-1}, a)|d(x_{2k-1}, Tx_{2k-1})| \\ &\quad + \gamma(Tx_{2k-1}, x_{2k-1}, a)|d(Tx_{2k-1}, x_{2k-1})| \\ &= \alpha(x_{2k}, x_{2k-1}, a)|d(x_{2k}, x_{2k+1})| \\ &\quad + \beta(x_{2k}, x_{2k-1}, a)|d(x_{2k-1}, x_{2k})| \\ &\quad + \gamma(x_{2k}, x_{2k-1}, a)|d(x_{2k}, x_{2k-1})| \\ &\leq \alpha(x_0, x_{2k-1}, a)|d(x_{2k}, x_{2k+1})| \\ &\quad + \beta(x_0, x_{2k-1}, a)|d(x_{2k-1}, x_{2k})| \\ &\quad + \gamma(x_0, x_{2k-1}, a)|d(x_{2k}, x_{2k-1})| \\ &\leq \alpha(x_0, x_1, a)|d(x_{2k}, x_{2k+1})| + \beta(x_0, x_1, a)|d(x_{2k-1}, x_{2k})| \\ &\quad + \gamma(x_0, x_1, a)|d(x_{2k}, x_{2k-1})| \end{aligned}$$

which yields that

$$|d(x_{2k+1}, x_{2k})| \leq \frac{\beta(x_0, x_1, a) + \gamma(x_0, x_1, a)}{1 - \alpha(x_0, x_1, a)} |d(x_{2k-1}, x_{2k})|.$$

Similarly, one can obtain

$$|d(x_{2k+2}, x_{2k+1})| \leq \frac{\beta(x_0, x_1, a) + \gamma(x_0, x_1, a)}{1 - \alpha(x_0, x_1, a)} |d(x_{2k}, x_{2k+1})|.$$

$$\text{Let } \mu = \frac{\beta(x_0, x_1, a) + \gamma(x_0, x_1, a)}{1 - \alpha(x_0, x_1, a)} < 1.$$

Since $\alpha(x_0, x_1, a) + \beta(x_0, x_1, a) + \gamma(x_0, x_1, a) < 1$, thus we have

$|d(x_{2k+2}, x_{2k+1})| \leq \mu |d(x_{2k}, x_{2k+1})|$, or in fact

$$|d(x_{n+1}, x_n)| \leq \mu |d(x_{n-1}, x_n)| \forall n \in N.$$

Now, from Lemma 2.2, we have $\{x_n\}$ is a Cauchy sequence in (X, d) .

By the completeness of X there exists $z \in X$ such that $x_n \rightarrow z$ as $n \rightarrow \infty$. Now we show that z is a fixed point of S . Now by (2.2) and Proposition 2.1, we have

$$\begin{aligned} d(z, Sz) &\lesssim d(z, Tx_{2n+1}) + d(Tx_{2n+1}, Sz) \\ &= d(z, x_{2n+2}) + d(Sz, Tx_{2n+1}) \\ &\lesssim d(z, x_{2n+2}) + \alpha(z, x_{2n+1}, a)d(z, Sz) \\ &\quad + \beta(z, x_{2n+1}, a)d(x_{2n+1}, Tx_{2n+1}) + \gamma(z, x_{2n+1}, a)d(z, x_{2n+1}) \\ &\lesssim d(z, x_{2n+2}) + \alpha(z, x_1, a)d(z, Sz) \\ &\quad + \beta(z, x_1, a)d(x_{2n+1}, x_{2n+2}) + \gamma(z, x_1, a)d(z, x_{2n+1}) \end{aligned}$$

which on letting $n \rightarrow \infty$, give rise $d(z, Sz) = 0 \Rightarrow Sz = z$. Now, we shall show that z is fixed point of T . By using inequality (2.2), we have

$$\begin{aligned} d(z, Tz) &\lesssim d(z, Sx_{2n}) + d(Sx_{2n}, Tz) \\ &= d(z, x_{2n+1}) + d(Sx_{2n}, Tz) \\ &\lesssim d(z, x_{2n+1}) + \alpha(x_{2n}, z, a)d(x_{2n}, Sx_{2n}) + \beta(x_{2n}, z, a)d(z, Tz) \\ &\quad + \gamma(x_{2n}, z, a)d(x_{2n}, z) \\ &\lesssim d(z, x_{2n+1}) + \alpha(x_0, z, a)d(x_{2n}, x_{2n+1}) + \beta(x_0, z, a)d(z, Tz) \\ &\quad + \gamma(x_0, z, a)d(x_{2n}, z) \end{aligned}$$

which on letting $n \rightarrow \infty$, we get $d(z, Tz) = 0$ and hence $Tz = z$.

This, implies that z is a common fixed point of S and T . Finally, we show the uniqueness. Suppose that there is $z^* \in X$ such that $z^* = Sz^* = Tz^*$. Then

$$\begin{aligned}
 d(z, z^*) &= d(Sz, Tz^*) \\
 &\lesssim \alpha(z, z^*, a)d(z, Sz) + \beta(z, z^*, a)d(z^*, Tz^*) \\
 &\quad + \gamma(z, z^*, a)d(z, z^*) \\
 &= \gamma(z, z^*, a)d(z, z^*).
 \end{aligned}$$

Therefore, we have

$$|d(z, z^*)| \leq \gamma(z, z^*, a)|d(z, z^*)|.$$

Since $\alpha(z, z^*, a) + \beta(z, z^*, a) + \gamma(z, z^*, a) < 1$, we have $|d(z, z^*)| = 0$. Thus $z = z^*$.

This concludes the theorem.

By choosing point dependent control function α, β, γ and mappings S and T suitably, one can deduce subsequent corollaries. Choosing $\alpha = 0$ and $\beta = 0$ in Theorem 2.3 results in following corollary.

Corollary 2.4. Let (X, d) be a complete complex valued metric space and $S, T: X \rightarrow X$ be mappings. If there exists mapping $\gamma: X \times X \times X \rightarrow [0, 1)$ such that for all $x, y \in X$:

$$\begin{aligned}
 \gamma(TSx, y, a) &\leq \gamma(x, y, a) \text{ and } \gamma(x, STy, a) \leq \gamma(x, y, a) \text{ satisfying} \\
 d(Sx, Ty) &\lesssim \gamma(x, y, a)d(x, y), \tag{2.6}
 \end{aligned}$$

for all $x, y \in X$ and for a fixed $a \in X$, then S and T have a unique common fixed point.

Opting $\gamma = 0$ in Theorem 2.3, we get the following observation.

Corollary 2.5. Let (X, d) be a complete complex valued metric space and $S, T: X \rightarrow X$ be mappings. If there exist mappings $\alpha, \beta: X \times X \times X \rightarrow [0, 1)$ such that for all $x, y \in X$:

$$\begin{aligned}
 \alpha(TSx, y, a) &\leq \alpha(x, y, a) \text{ and } \alpha(x, STy, a) \leq \alpha(x, y, a), \\
 \beta(TSx, y, a) &\leq \beta(x, y, a) \text{ and } \beta(x, STy, a) \leq \beta(x, y, a) \text{ and}
 \end{aligned}$$

$$\alpha(x, y, a) + \beta(x, y, a) < 1,$$

also satisfying

$$d(Sx, Ty) \lesssim \alpha(x, y, a)d(x, Sx) + \beta(x, y, a)d(y, Ty), \quad (2.7)$$

then S and T have a unique common fixed point.

Taking $S = T$ in Theorem 2.3, we get the another result.

Corollary 2.6. Let (X, d) be a complete complex valued metric space and $T: X \rightarrow X$ be mapping. If there exist mappings $\alpha, \beta, \gamma: X \times X \times X \rightarrow [0, 1)$ such that for all $x, y \in X$ and for fixed $a \in X$,

$$\begin{aligned} \text{a) } & \alpha(Tx, y, a) \leq \alpha(x, y, a) \text{ and } \alpha(x, Ty, a) \leq \alpha(x, y, a), \\ & \beta(Tx, y, a) \leq \beta(x, y, a) \text{ and } \beta(x, Ty, a) \leq \beta(x, y, a), \\ & \gamma(Tx, y, a) \leq \gamma(x, y, a) \text{ and } \gamma(x, Ty, a) \leq \gamma(x, y, a), \\ \text{b) } & d(Tx, Ty) \lesssim \alpha(x, y, a)d(x, Tx) + \beta(x, y, a)d(y, Ty) \\ & \quad + \gamma(x, y, a)d(x, y), \\ \text{c) } & \alpha(x, y, a) + \beta(x, y, a) + \gamma(x, y, a) < 1, \end{aligned} \quad (2.8)$$

then T has a unique fixed point.

Corollary 2.7. In Corollary 2.6, if we define mappings $\alpha, \beta, \gamma: X \times X \times X \rightarrow [0, 1)$ such that $\alpha(x, y, a) = \alpha(x, y)$, $\beta(x, y, a) = \beta(x, y)$ and $\gamma(x, y, a) = \gamma(x, y)$, then for all $x, y \in X$,

$$\begin{aligned} \text{a) } & \alpha(Tx, y) \leq \alpha(x, y) \text{ and } \alpha(x, Ty) \leq \alpha(x, y), \\ & \beta(Tx, y) \leq \beta(x, y) \text{ and } \beta(x, Ty) \leq \beta(x, y), \\ & \gamma(Tx, y) \leq \gamma(x, y) \text{ and } \gamma(x, Ty) \leq \gamma(x, y), \\ \text{b) } & d(Tx, Ty) \lesssim \alpha(x, y)d(x, Tx) + \beta(x, y)d(y, Ty) + \gamma(x, y)d(x, y), \\ \text{c) } & \alpha(x, y) + \beta(x, y) + \gamma(x, y) < 1, \end{aligned} \quad (2.9)$$

then T has a unique fixed point.

Corollary 2.8. The conclusion of Theorem 2.3 remains true if we replace inequality (2.2) by the following:

$$d(Sx, Ty) \lesssim \alpha(x, y, a)d(x, Sx) + \beta(x, y, a)d(y, Ty) + \gamma(x, y, a)d(x, y) + \frac{\delta(x, y, a)d(x, Sx)d(y, Ty)}{1+d(x, y)} \quad (2.10)$$

holds for all $x, y \in X$, where control functions are satisfying the condition $\alpha(x, y, a) + \beta(x, y, a) + \gamma(x, y, a) + \delta(x, y, a) < 1$.

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