NUMERICAL SIMULATION OF A FREE SURFACE FLOW DUE TO A POINT SOURCE

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ABSTRACT

The Boundary Element Method (BEM) has become an efficient and popular alternative to the Finite Element Method (FEM) because of its ability of reducing a Boundary Value Problem (BVP) for a linear Partial Differential Equation (PDE) defined in a domain to an integral equation defined on the boundary leading to a simplified discretization process with boundary elements only. The main requirement for the reduction of the PDE to a Boundary Integral Equation (BIE) is that a fundamental solution of the PDE must be available. For many PDEs with constant coefficients, the fundamental solutions are well known. In this paper, we discuss the BEM used in the solution of a BIE formed from the free surface flow due to a point source. The solution procedure involves the method of fundamental solutions with the adoption of the Standard Gaussian Quadrature method. A numerical example using different number of nodal points is presented and the obtained results are compared to check the efficiency and good precision of the adopted method and the effect of varying the number of nodal points in the final results.

KEYWORDS: Boundary Integral Equation, Quadrature Technique (QT)

INTRODUCTION

The origin of numerical implementation of boundary integral equations can be traced back to the availability of electronic computers. The full emergence of the numerical technique known as boundary element method occurred in the late 1970s and was first used by Brebbia and Dominguez (1977) who realized the analogy between the discretization process for the Boundary Integral Method (BIM) and that for the already established Finite Element Method (FEM). Only the boundary of the solution domain has to be discretized into elements in the implementation of the method. The term Boundary Element Method denotes any method for the approximate numerical solution of boundary integral equations. The approximate solution of the boundary value problem obtained by BEM is an exact solution of the differential equation in the domain and is parameterized by a finite set of parameters on the boundary. The BEM is a convenient method for solving PDEs in that it requires discretization only on the boundary of the domain (Brebbia et. al. 1984). Using BEM it is possible to obtain better accuracy with a fewer number of elements when it is compared to FEM (LaForce T., 2006). On the other hand, BEM requires a fundamental solution for the governing solution; therefore it is mainly applicable for linear differential equations (Antes H., 2010). It is a numerical technique to solve BVPs represented by linear

PDEs (Brebbia et. al. 1980). The method has become a widely used numerical tool in statics and dynamics (Gaul and Fiedler, 1997). A review about the efforts in dynamics is published by Beskos (1987; 1997)

The BEM is derived through the discretization of an integral equation that is mathematically equivalent to the original PDE. The essential formulation of the PDE that underlies the BEM consists of an integral equation that is defined on the boundary of the domain and an integral that relates the boundary solution to the solution at the points in the domain. The former is termed a boundary integral equation and the BEM is often referred to as the numerical boundary integral method.

Wu (1985) argued that the BEM has several advantages over other numerical methods which justify its use for many practical problems – even in cases where domain integration is required. He argued that for problems such as flow problems a wide range of phenomena are described by the problem. For this reason, accurate description of the boundary conditions is vital for solution accuracy. The BEM generates a formulation involving both the dependent variable u and the flux q. This allows flux boundary conditions to be applied directly which cannot be
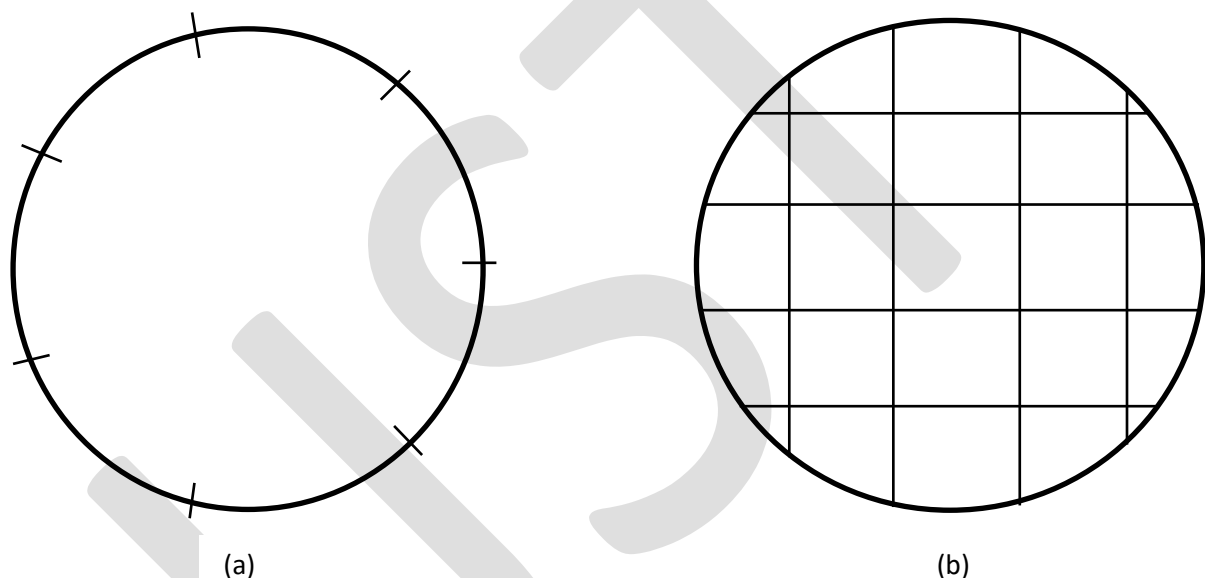

Figure 1: Schematic illustration of a boundary element (a) and a finite element (b)

achieved in either the FEM or Finite Difference Methods (FDM). Another advantage of BEM over other numerical methods is that it allows an explicit expression for the solution at an internal point. This allows a problem to be subdivided into a number of zones for which the BEM can be applied individually. The main advantage is that it replaces the original problem with an integral equation defined on the boundary of the solution domain. For the case of a homogeneous PDE, it only requires the discretization on the boundary of the domain (Kythe, 1995). The BEM compliments the Finite Element Method (FEM) to solve the boundary value problems (BVP). The main difference between the BEM and FEM is that the FEM is a regional method which discretizes the whole region of interest.

If the region of interest is not regular and automatic discretization techniques cannot be used, an artificial method must be used to discretize the region. This therefore implies that the preparation and input of data used to model a BVP into a computer may become a very complex task. The BEM divides only the boundary and this diminishes the dimensionality of the problem and a 2-D problem becomes a 1-D problem and hence input of

data to model the problem into a computer becomes a less complex task as the number of resulting algebraic equations involved is significantly reduced.

The numerical procedure therefore first involves discretizing the surface $\partial\Omega$ into some set of elements as shown in Figure 1.

MATHEMATICAL MODEL

The Laplace equation governing potential problems in a 2D domain Ω is given by:

$$\nabla^2\phi(\bar{x}) \;=\; 0, \quad \forall \bar{x} \;\in\; \Omega$$

(1)

under the boundary conditions:

$$\phi(\bar{x}) \;=\; \bar{\phi}(\bar{x}) \quad \forall x \;\in\; \Omega_1$$

(2)

$$q(\bar{x}) \;\equiv\; \frac{\partial\phi}{\partial n}(\bar{x})$$
$$=\; \bar{q}(\bar{x}), \quad \forall x \;\in\; \Omega_2$$

(3)

where ϕ is the potential field in the domain Ω, $\Omega_1 \cup \Omega_2$ the boundary of Ω, n the outward normal and $\bar{\phi}$ and $\bar{q}$ indicate given values on the boundary. The solution of the Boundary Value Problem (BVP) described by eqns (1 – 3) can be written in the following representation integral (Banerjee, 1994).

$$\phi(\bar{x}) \;=\; \int_{\Omega}\Big[G(\bar{x},\bar{\xi})q(\bar{\xi}) \;-\; F(\bar{x},\bar{\xi})\phi(\bar{\xi})\Big]d\Omega(\xi), \qquad \forall x \;\in\; \Omega$$

(4)

where

$$G(\bar{x},\bar{\xi}) \;=\; \frac{-1}{2\pi}\ln r$$

(5)

$$F(\bar{x},\bar{\xi}) \;=\; \frac{\partial G}{\partial n}$$
$$=\; \frac{1}{2\pi r}\frac{\partial r}{\partial n}$$

(6)

with r being the distance between points x and ξ.

Letting $\xi \rightarrow \Omega$ we obtain the following Boundary Integral Equation (BIE):

$$C(\bar{x})\phi(\bar{x}) \;=\; \int_{\Omega}\Big[G(\bar{x},\bar{\xi})q(\bar{\xi}) - F(\bar{x},\bar{\xi})\phi(\bar{\xi})\Big]d\Omega(\bar{\xi}) \qquad \forall \; \bar{x} \;\in\; \Omega$$

(7)

where

$$\phi(\bar{\xi}) \quad = \quad \frac{M}{2\pi}$$

$$q(\bar{\xi}) \quad = \quad \frac{M}{2\pi}\frac{\bar{r}}{r^2}$$

(8)

The second term in the right hand side of (7) is a singular integral of the Cauchy-Principal Value (CPV) type and the coefficient:

$$C(\bar{x}) \quad = \quad - \int_{\partial\Omega} F(\bar{\xi},\bar{x})dS(\bar{x})$$

(9)

which is also a CPV integral.

Substituting (9) in (7), we obtain the following weakly singular form of the BIE for potential problems (Liu Y.J et. al., 1991; 1999)

$$\int_{\Omega} F(\bar{x},\bar{\xi})\left[\phi(\bar{\xi})-\phi(\bar{x})\right]d\Omega(\bar{\xi}) \quad = \quad \int_{\Omega} G(\bar{x},\bar{\xi})q(\bar{\xi})d\Omega(\bar{\xi}) \qquad \forall \quad \bar{x} \quad \in \quad \Omega$$

(10)

In which no singular integrals exist.

In discretization, we use the constant boundary elements, that is, dividing the boundary S into N line segments (elements) and placing one node on each element as shown in figure 2. The following discretized equation of BIE (7) is obtained (Brebbia, 1989; Banerjee, 1994; Kane, 1994)

$$c_i\phi_i \quad + \quad \sum_{j=1}^{N}F_{ij}\phi_j \quad = \quad \sum_{j=1}^{N}G_{ij}q_j \qquad \text{for i = 1, 2, ... , N}$$

(11)

where ϕ_i and q_j (j = 1, 2, ... , N) are the nodal values of ϕ and q on element $\partial\Omega_j$ respectively and the coefficients are given by (Nissaya and Wattana, 2011)

$$F_{ij} \quad = \quad \begin{cases} f_1^{ij} \quad + \quad f_2^{iN} & for \; j=1 \\ f_1^{ij} \quad + \quad f_2^{i(j-1)} & for \; j=2,3,....,N \end{cases}$$

(12)

$$G_{ij} \quad = \quad \begin{cases} g_1^{ij} \quad + \quad g_2^{iN} & for \; j=1 \\ g_1^{ij} \quad + \quad g_2^{i(j-1)} & for \; j=2,3,....,N \end{cases}$$

(13)

$$g_1^{ij} \quad = \quad \frac{l_j}{2} \int_{-1}^{1} G\phi_1(\xi)d\xi$$

$$g_2^{i(j-1)} \quad = \quad \frac{l_{j-1}}{2} \int_{-1}^{1} G\phi_2(\xi)d\xi$$

(14)

(15)

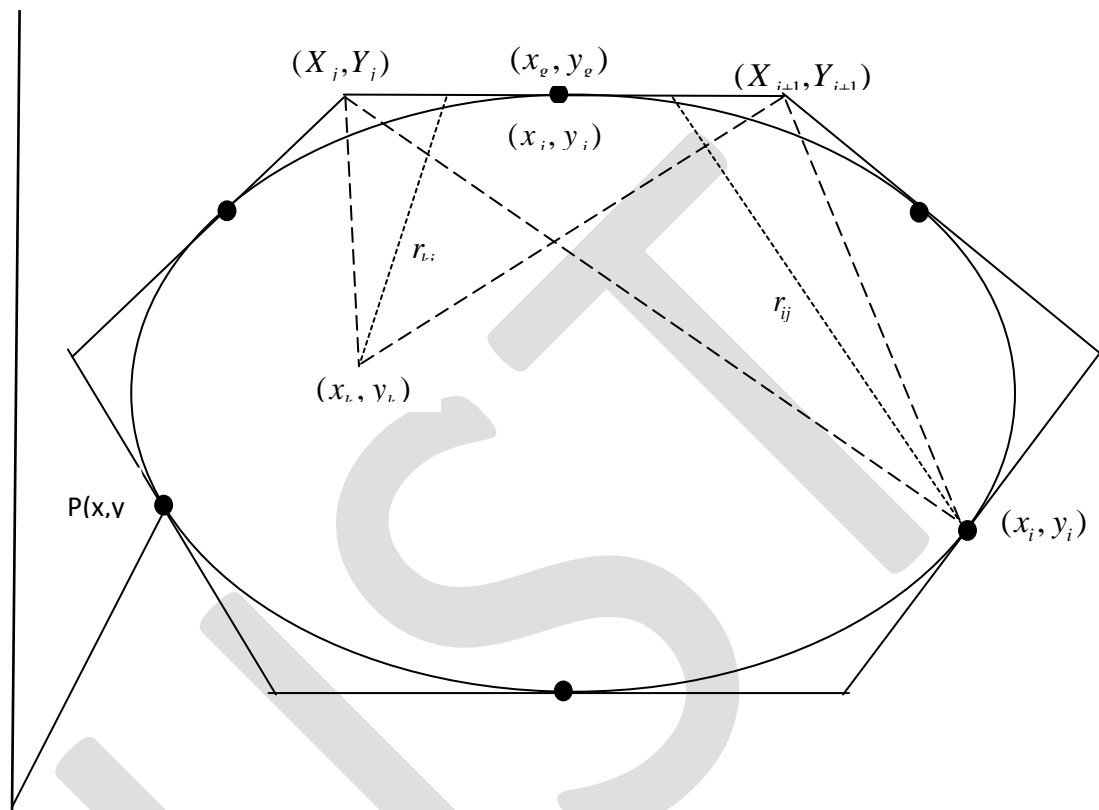**Figure 2: Geometry of the Boundary Element Method**

$$f_1^{ij} \quad = \quad \frac{l_j}{2} \int_{-1}^{1} q^*\phi_1(\xi)d\xi$$

$$f_2^{i(j-1)} \quad = \quad \frac{l_{j-1}}{2} \int_{-1}^{1} q^*\phi_2(\xi)d\xi$$

(16)

(17)

$$\phi_1 \quad = \quad \frac{1}{2}(1-\xi) \qquad \phi_2 \quad = \quad \frac{1}{2}(1+\xi)$$

(18)

where ξ is the dimensionless co-ordinate and $\phi_1 \quad and \quad \phi_2$ are the usual Lagrange interpolation polynomials

and

$$r \quad = \quad \sqrt{\left|x(\xi)-x_i\right|^2 + \left|y(\xi)-y_i\right|^2}$$

(19)

The integrals in equations (14) to (17) go from – 1 to 1 in a procedure called standard Gaussian Quadrature. Integrals with different intervals can be transformed to standard Gaussian Quadrature with a coordinate transformation by the formula:

$$\int_a^b f(x)dx \;=\; \frac{(b-a)}{2}\int_{-1}^1 f\left[\frac{b+a}{2}+\left(\frac{b-a}{2}\right)t\right]dt$$

(20)

After the boundary is discretized into elements and within each element some polynomial shape functions are adopted to approximate the variation of ϕ and $\dfrac{\partial\phi}{\partial n}\equiv q$, the boundary integrals in equation (7) can be discretized so that the nodal values of ϕ and q are taken out of the integral signs. Consequently, a linear algebraic system (21) is formed upon placing the source point at each boundary node and carrying out the integral within each boundary element.

$$F\phi \;=\; Gq$$

(21)

where

$$F \;=\; \begin{cases} F & i \;\neq\; j \\ F+c_i & i \;=\; j \end{cases}$$

(22)

The matrices **F** and **G** are of the conventional sense (Brebbia et. al., 1984) and the vectors ϕ and q contain the values of ϕ and q at each boundary node. Assuming mixed boundary conditions, the part Ω_1 of the boundary on which ϕ is described and the part Ω_2 on which q is described are discretized into N_1 and N_2 linear elements respectively such that $\Omega_1\cup\Omega_2 \;=\; \Omega,$

$$\Omega_1\cap\Omega_2 \;=\; \{\ \} \text{ and }$$

$$N_1+N_2 \;=\; N$$

Separating the unknown from the known quantities of (21), we write after partitioning the matrices in the following way:

$$\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \;=\; \begin{bmatrix} G_1 & G_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

(23)

where u_1 and q_2 denote the prescribed quantities on Ω_1 and Ω_2 respectively, q_1 and u_2 denote the corresponding unknown ones. Moving all the unknowns to the left side of the equation, we obtain the system:

$$Ax \;=\; B$$

(24)

where

$$A = \begin{bmatrix} F_2 & -G_1 \end{bmatrix},$$

$$x = \begin{bmatrix} \phi_2 \\ q_1 \end{bmatrix},$$

$$B = -F_1\phi_1 + G_2 q_2$$

(25)

A is a square matrix with dimensions $N \times N$. B and x are vectors with dimension N. We can now find

q and ϕ at every point on the boundary by using $x = A^{-1}B$. Knowing all the boundary quantities on

$\partial\Omega$ therefore implies that the solution ϕ can be computed at any point $P(x, y)$ in the domain Ω by the

equation (8) as follows:

$$c_i\phi_i = \sum_{j=1}^{N}\int_{\partial\Omega_j} Gq\,dS - \sum_{j=1}^{N}\int_{\partial\Omega_j}\phi q^*\,dS$$

(26)

where

$$G = \frac{-1}{2\pi}\ln r$$

$$c_i = \frac{\alpha_i}{2\pi}$$

$$q^* = \frac{\partial G}{\partial n}$$

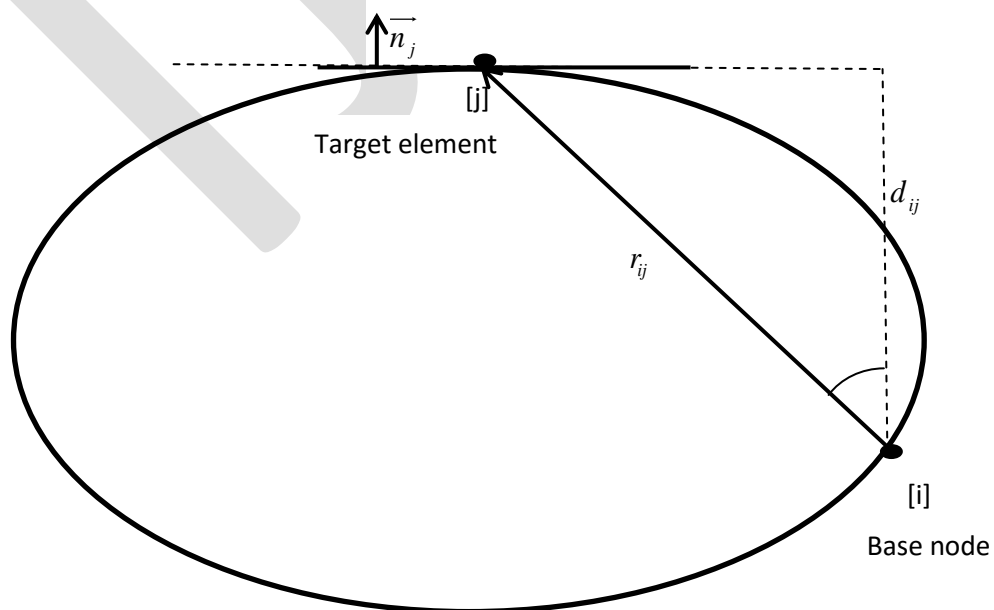$$= \frac{-1}{2\pi}\frac{\partial \ln r}{\partial n}$$

(27)

Figure 3: Distance from the node i to the element [j]

From (27), equation (26) can be rewritten as:

$$\alpha_i \phi_i \quad = \quad \sum_{j=1}^{N} \overline{F}_{ij} \phi_j \quad + \quad \sum_{j=1}^{N} \overline{G}_{ij} q_j$$

(28)

where

$$\overline{F}_{ij} \quad = \quad \int_{[j]} \frac{\partial}{\partial n} \left(\ln r_{ij} \right) dS$$

(29)

$$\overline{G}_{ij} \quad = \quad - \int_{[j]} \ln r_{ij} dS$$

(30)

as shown in the Fig. 3. It can be seen that

$$r_{ij}.\overline{\mathbf{n}}_j \quad = \quad \left| r_{ij} \right| \left| \overline{\mathbf{n}}_j \right| \cos \theta$$

$$= \quad \left| r_{ij} \right|.1.\frac{1}{\left| r_{ij} \right|} \mathbf{d}_{ij}$$

$$= \quad \pm \mathbf{d}_{ij}$$

(31)

Where d_{ij} is the perpendicular distance from node i to target element j. Hence:

$$\frac{\partial}{\partial n}\left(\ln r_{ij} \right) \quad = \quad grad\left(\ln r_{ij} \right).\overline{n}_j$$

(32)

Let $A_{ij} \quad = \quad \frac{1}{2} d_{ij} l_j$ be the area of the triangle which has element j as its base and node i as a vertex. To

perform the integrals (29) and (30), we use the N-point Gaussian quadrature over element j, with length l_j so

that:

$$\overline{F}_{ij} \quad = \quad \frac{d_{ij} l_j}{2} \int_{-1}^{1} \frac{1}{r_{ij}^2} dt \quad \approx \quad A_{ij} \sum_{g=1}^{P} w_g \frac{1}{(r_{ij}^2)_g}$$

(33)

$$\overline{G}_{ij} \quad = \quad \frac{-l_j}{4} \int_{-1}^{1} \ln r_{ij} dt \quad \approx \quad \frac{-l_j}{4} \sum_{g=1}^{P} w_g \ln(r_{ij})_g$$

(34)

where P is the number of Gauss integration points.

If $i \quad = \quad j$, then $r_{ii} \quad and \quad \overline{n}_i$ are orthogonal, so that $r_{ii} \circ \overline{n}_i \quad = \quad 0$. Hence:

$$\overline{F}_{ii} \quad = \quad 0 \qquad i = 1, 2, 3, \dots, N$$

(35)

For $\overline{G}_{ii}$, integrate analytically to obtain:

$$\overline{G}_{ii} \quad = \quad l_i \left[1 - \ln\left(\frac{l_i}{2}\right) \right]$$

(36)

Since $\phi \equiv 1$ is harmonic in Ω with $\phi \equiv 1 \quad and \quad q = 0$ on $\partial\Omega$, (28) yields

$$\alpha_i \quad = \quad \sum_{j=1}^{N} \overline{F}_{ij}$$

(37)

When (21) has been solved, the values of $\phi_i \quad and \quad q_i$ at all nodal points are known.

In a similar manner of approximating $\overline{F}_{ij} \quad and \quad \overline{G}_{ij}$ for nodal points we can obtain $\overline{F}_{kj} \quad and \quad \overline{G}_{kj}$ for internal points as follows:

$$\overline{F}_{kj} \quad \approx \quad A_{kj} \sum_{g=1}^{P} w_g \frac{1}{(r_{kj})_g^2}$$

(38)

$$\overline{G}_{kj} \quad \approx \quad \frac{-l_j}{4} \sum_{g=1}^{P} w_g \ln(r_{kj})_g$$

(39)

The solution for internal points is therefore arrived at as:

$$\phi_k \quad = \quad \frac{1}{2\pi} \left(\sum_{j=1}^{N} \overline{F}_{kj} \phi_j + \sum_{j=1}^{N} \overline{G}_{kj} q_j \right)$$

(40)

NUMERICAL ANALYSIS

It is clear from Fig. 2 that if P(x,y) is an arbitrary point then the relationships between coordinates and the angle θ are as follows:

$$x \quad = \quad |OP|\cos\theta$$
$$y \quad = \quad |OP|\sin\theta$$

(41)

Suppose:

(x_i, y_i) be the coordinate of node i

(x_j, y_j) be the coordinate of node j of element j

(X_j, Y_j) be the coordinate of end point of j

(x_k, y_k) be the coordinate of internal point k

(x_g, y_g) be the coordinate of Gauss quadrature point

The areas A_{ij} and A_{kj} can be computed as:

$$A_{ij} \quad = \quad \frac{1}{2}\left|x_i y_j + X_j Y_{j+1} + X_{j+1} y_i - X_j y_i - X_{j+1} Y_j - x_i Y_{j+1}\right|$$

$$(42)$$

$$A_{kj} \quad = \quad \frac{1}{2}\left|x_k y_j + X_j Y_{j+1} + X_{j+1} y_k - X_j y_k - X_{j+1} Y_j - x_k Y_{j+1}\right|$$

$$(43)$$

In the Gaussian quadrature the integration points have specified positions called Gauss points. These points are weighted with a weighting factor. The Gaussian quadrature provides an exact solution for integrals up to order 2n – 1 with n as the number of Gauss points. The integration over each element is obtained by using the Gauss – Legendre N-point method which gives:

$$x \quad = \quad x_j \quad + \quad \frac{1}{2}(X_{j+1} - X_j)t$$

$$y \quad = \quad y_j \quad + \quad \frac{1}{2}(Y_{j+1} - Y_j)t$$

$$(44)$$

where t are the roots of the n$^{\text{th}}$ degree Lagrange Polynomial

$$P_n(t) \quad = \quad \frac{1}{2^n n!}\frac{d^n}{dt^n}\left(t^2 - 1\right)^n \quad \text{in (-1, 1).}$$

$$(45)$$

This therefore implies that in the element j we obtain the Gaussian quadrature points as

$$x_g \quad = \quad x_j \quad + \quad \frac{1}{2}(X_{j+1} - X_j)t_g$$

$$y_g \quad = \quad y_j \quad + \quad \frac{1}{2}(Y_{j+1} - Y_j)t_g$$

$$(46)$$

where the value of t and w (from tables of weighting factors and function arguments used in Gauss quadrature formulae) are:

$$t_g: \quad t_1 = -0.77459667, \quad t_2 = 0.0, \quad t_3 = 0.77459667$$
$$w_g: \quad w_1 = 0.55555555, \quad w_2 = 0.88888889, \quad w_3 = w_1$$

$$(47)$$

Hence

$$\left(r_{ij}^2\right)_g \quad = \quad (x_i - x_g)^2 \quad + \quad (y_i - y_g)^2$$

$$\left(r_{kj}^2\right)_g \quad = \quad (x_k - x_g)^2 \quad + \quad (y_k - y_g)^2$$

$$(48)$$

To construct the whole system of linear equations of all elements, we first obtain $\overline{F}_{ij}$ by using (33) and (35) as:

$$\overline{F}_{ij} \;=\; \begin{cases} 0 \quad,\quad i=j \\[2mm] A_{ij}\displaystyle\sum_{g=1}^{3}\left[w_g / (x_g - x_i)^2 + (y_g - y_i)^2 \right] \;,\quad i \neq j \end{cases}$$

(49)

where w_g are Gaussian weights given by the formula:

$$w_g \;=\; \frac{2(1 - t_g^{\,2})}{n^2 \left(P_{n-1}(t_g) \right)^2}$$

(50)

Then

$$F_{ij} \;=\; \begin{cases} -\displaystyle\sum_{j=1}^{N}\overline{F}_{ij}, \quad i=j \\[3mm] \overline{F}_{ij}, \quad i \neq j \end{cases}$$

(51)

Constructing G_{ij} using equations (34) and (36) we have:

$$G_{ij} \;=\; \begin{cases} l_j(1 - \ln(l_j/2)), & i=j \\[3mm] \dfrac{l_j}{4}\displaystyle\sum_{g=1}^{3} w_g \ln\left[(x_g - x_i)^2 + (y_g - y_i)^2 \right], & i \neq j \end{cases}$$

(52)

Adding the Dirichlet and Neumann conditions of the boundary points so that we have $\phi_j \quad and \quad q_j$ in hand, we obtain the system of linear equations as:

$$F\phi + Gq \;=\; b$$

(53)

By setting $b_{ij} \;=\; 0$ for $j = 1, 2, 3, \ldots, N$ by (49), we obtain the solutions $\overline{\phi}_j \quad and \quad \overline{q}_j$. From (33),

$$\overline{\overline{F}}_{kj} \;=\; A_{kj}\displaystyle\sum_{g=1}^{3}\left[w_g / (x_g - x_k)^2 + (y_g - y_k)^2 \right]$$

(54)

$$\overline{\overline{G}}_{kj} \;=\; \frac{-l_j}{4}\displaystyle\sum_{g=1}^{3} w_g \ln\left[(x_g - x_k)^2 + (y_g - y_k)^2 \right]$$

(55)

The interior solution will thus be:

$$\phi_k \;\approx\; \frac{1}{2\pi}\left[\displaystyle\sum_{j=1}^{N}\overline{\overline{F}}_{kj}\overline{\phi}_j + \sum_{j=1}^{N}\overline{\overline{G}}_{kj}\overline{q}_j \right] \quad ,k = 1, 2, \ldots, m$$

(56)

NUMERICAL EXAMPLE

Using a circle of radius 2 units the method is illustrated using different number of internal points and 40 nodal points for each to compare the outcome and also find the error from each of the applications. The outcome is shown in table 1 and the output for each also worked out. The comparison of the different outputs and error differences is also shown. The problem can be stated thus:

$$\Delta^2 u = 0 \qquad \text{in} \qquad x^2 + y^2 \ < \ 4$$

Subject to the boundary conditions

$$u \ = \ x + y \quad on \quad x^2 + y^2 = 4, \quad x \le 4$$

$$q \ = \ \frac{1}{2}(x + y) \quad on \quad x^2 + y^2 = 4, \quad x > 4$$

The conditions form both the Dirichlet and Neumann Boundary Conditions

CONCLUSION

With the BEM, we can approximate the solution of a PDE by looking at the solution to the PDE on the boundary and then use that information to find the solution inside the domain. A method such as the BEM would best be implemented with a tool such as MATLAB that can perform matrix computations and numerical integration efficiently and produce graphical output. From the output it can be seen that as the number of internal points is increased there is a significant improvement in the accuracy of the numerical result as the error decreases to a point where it is almost non-existent when the number of internal points is 64 (see figures and the table 1 below).

Table1: The internal solutions of 8, 16, 32, 64 elements with 40 internal points

Solution by N nodal points				Exact solution	Errors in each case			
8	16	32	64	u = xi + yi	8	16	32	64
Uk	Uk	uk	uk	Xk	Uk	Uk	uk	Uk
5.4977	3.3250	2.3508	2.0128	1.9500	3.5477	1.3750	0.4008	0.0628
2.5016	1.9746	1.8530	1.8476	1.8500	0.6516	0.1246	0.0030	0.0024
1.9070	1.7435	1.7398	1.7472	1.7500	0.1570	0.0065	0.0102	0.0028
1.6452	1.6199	1.6391	1.6473	1.6500	0.0048	0.0301	0.0109	0.0027
1.4827	1.5153	1.5394	1.5474	1.5500	0.0673	0.0347	0.0106	0.0026
1.3578	1.4152	1.4398	1.4476	1.4500	0.0922	0.0348	0.0102	0.0024
1.2489	1.3164	1.3403	1.3477	1.3500	0.1011	0.0336	0.0097	0.0023
1.1475	1.2180	1.2407	1.2478	1.2500	0.1025	0.0320	0.0093	0.0022
1.0497	1.1198	1.1412	1.1479	1.1500	0.1003	0.0302	0.0088	0.0021
0.9539	1.0215	1.0417	1.0480	1.0500	0.0961	0.0285	0.0083	0.0020
0.8591	0.9233	0.9421	0.9481	0.9500	0.0909	0.0267	0.0079	0.0019
0.7649	0.8251	0.8426	0.8482	0.8500	0.0851	0.0249	0.0074	0.0018
0.6709	0.7268	0.7431	0.7483	0.7500	0.0791	0.0232	0.0069	0.0017
0.5769	0.6285	0.6435	0.6485	0.6500	0.0731	0.0215	0.0065	0.0015
0.4829	0.5302	0.5440	0.5486	0.5500	0.0671	0.0198	0.0060	0.0014
0.3888	0.4318	0.4444	0.4487	0.4500	0.0612	0.0182	0.0056	0.0013
0.2946	0.3334	0.3448	0.3488	0.3500	0.0554	0.0166	0.0052	0.0012
0.2003	0.2350	0.2453	0.2489	0.2500	0.0497	0.0150	0.0047	0.0011
0.1059	0.1366	0.1457	0.1490	0.1500	0.0441	0.0134	0.0043	0.0010
0.0113	0.0381	0.0461	0.0491	0.0500	0.0387	0.0119	0.0039	0.0009
-0.0834	-0.0604	-0.0535	-0.0508	-0.0500	0.0334	0.0104	0.0035	0.0008
-0.1782	-0.1590	-0.1530	-0.1507	-0.1500	0.0282	0.0090	0.0030	0.0007
-0.2732	-0.2575	-0.2526	-0.2506	-0.2500	0.0232	0.0075	0.0026	0.0006
-0.3683	-0.3561	-0.3523	-0.3505	-0.3500	0.0183	0.0061	0.0023	0.0005

|								
-0.4635	-0.4548	-0.4519	-0.4504	-0.4500	0.0135	0.0048	0.0019	0.0004
-0.5589	-0.5535	-0.5515	-0.5504	-0.5500	0.0089	0.0035	0.0015	0.0004
-0.6544	-0.6522	-0.6511	-0.6503	-0.6500	0.0044	0.0022	0.0011	0.0003
-0.7501	-0.7510	-0.7508	-0.7502	-0.7500	0.0001	0.0010	0.0008	0.0002
-0.8461	-0.8497	-0.8504	-0.8501	-0.8500	0.0039	0.0003	0.0004	0.0001
-0.9426	-0.9486	-0.9501	-0.9500	-0.9500	0.0074	0.0014	0.0001	0.0000
-1.0398	-1.0474	-1.0497	-1.0500	-1.0500	0.0102	0.0026	0.0003	0.0000
-1.1383	-1.1463	-1.1494	-1.1499	-1.1500	0.0117	0.0037	0.0006	0.0001
-1.2390	-1.2453	-1.2491	-1.2498	-1.2500	0.0110	0.0047	0.0009	0.0002
-1.3438	-1.3444	-1.3488	-1.3497	-1.3500	0.0062	0.0056	0.0012	0.0003
-1.4567	-1.4440	-1.4485	-1.4497	-1.4500	0.0067	0.0060	0.0015	0.0003
-1.5866	-1.5450	-1.5482	-1.5496	-1.5500	0.0366	0.0050	0.0018	0.0004
-1.7566	-1.6505	-1.6481	-1.6495	-1.6500	0.1066	0.0005	0.0019	0.0005
-2.0321	-1.7754	-1.7490	-1.7495	-1.7500	0.2821	0.0254	0.0010	0.0005
-2.6610	-2.0097	-1.8625	-1.8499	-1.8500	0.8110	0.1597	0.0125	0.0001
-5.8405	-3.3823	-2.3625	-2.0152	-1.9500	3.8905	1.4323	0.4125	0.0652
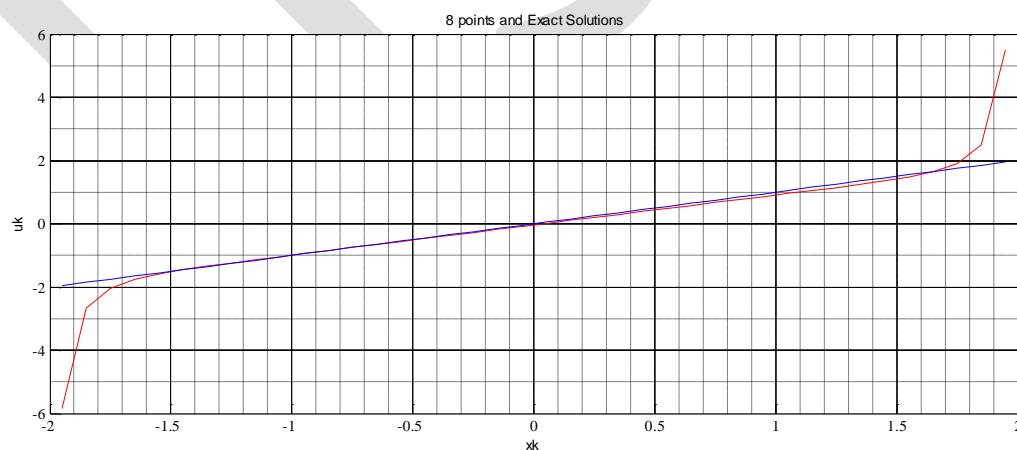
Figure 4: Comparison of Numerical solution using 8 Nodal Points with the Exact Solution
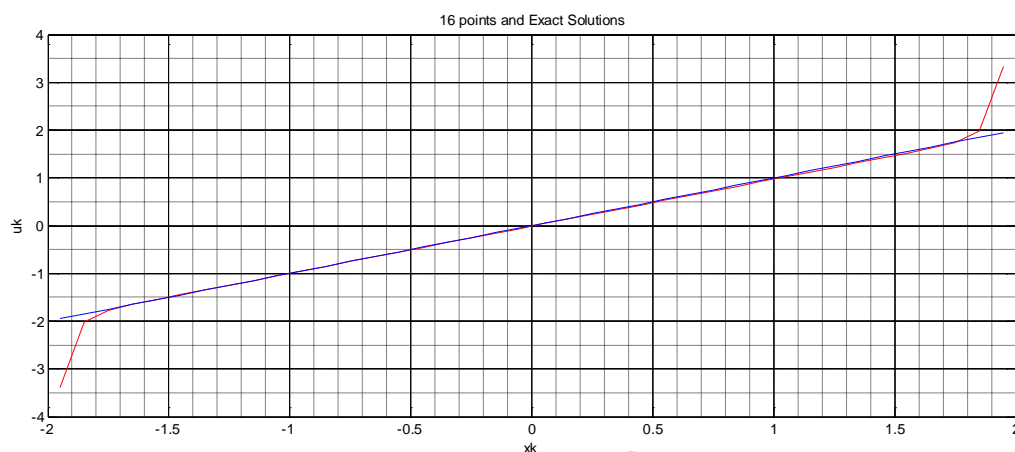
Figure 5: Comparison of Numerical solution using 16 Nodal Points with the Exact Solution

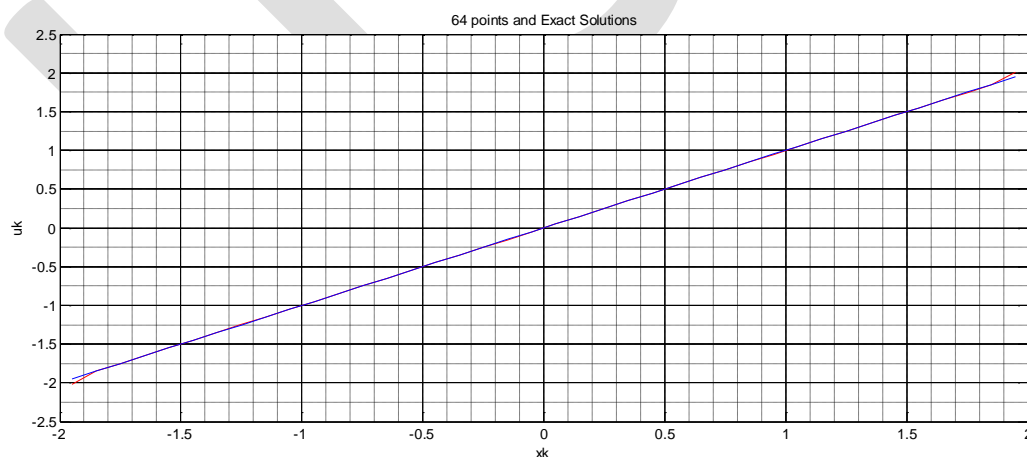**Figure 6: Comparison of Numerical solution using 32 Nodal Points with the Exact Solution**

Figure 7: Comparison of Numerical solution using 64 Nodal Points with the Exact Solution

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