A BOUNDARY INTEGRAL EQUATION APPROACH TO A FREE SURFACE FLOW DUE TO A POINT SOURCE

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ABSTRACT

Boundary Integral Equations (BIE) have been used to create effective methods for solving elliptic partial differential equations. Of primary importance is choosing the appropriate boundary representation for the solution such that the resulting integral equation is well – conditioned and solvable. The BIE is used as the starting point for developing a boundary element technique to solve problems in several fields of applied mathematics such as fluid mechanics. For problems with known Green's function, an integral equation formulation leads to powerful numerical approximation techniques. When a point source in placed in a body of fluid, the surface of the fluid will be deformed and may (or may not) exhibit a train of water waves. Mathematically, this is a nonlinear free surface flow problem which can be reformulated using Green's function as an integral equation. In this study, the boundary integral equation is formulated based on the Laplace equation and the application of the fundamental solution and Green's identities.

KEYWORDS: Boundary integral equation, Freespace Green's function, Laplace equation

INTRODUCTION

The solution of Partial Differential Equations (PDE) describing a wide range of physical problems in a domain Ω can be reduced to the solution of corresponding boundary integral equations (BIE) on a boundary C. The solution of boundary integral equations leads to the retrieval of unknown boundary values of the functions and/or the derivatives of these functions that occur in the original differential equation. The BIE can rarely be solved analytically. One of the most used techniques to their solution is the Boundary Element Method (BEM). In order to solve for the unknown surface data, the surface must be subdivided into segments and, as a results, the boundary integral equations are approximated by a system of algebraic equations. The boundary elements have one less dimension than the body being analyzed. That is, the boundary of a two-dimensional problem is surrounded by one-dimensional elements, while the surface of a three-dimensional solid is paved with two-dimensional elements. Consequently, boundary element analysis can be very efficient, particularly when the boundary quantities are of primary interest. BEM solutions have been found to be quite accurate, especially when the domain is infinite or semi-infinite.

There are two basic approaches leading to the formulation of BIE. The first approach is called direct formulation or “method of Green’s formula” and it leads to the construction of integral equations which contain, as unknown functions those functions, which stand in original differential equations. The second approach, called indirect formulation, leads to the integral equations which contain as unknown functions single layer potential densities and double layer potential densities, from which the searched functions, standing in the differential equations, must only be computed. While the theoretical basis of the direct formulation is the fundamental solution of a differential equation, which is used together with Green’s formulae, the basis of the indirect formulation is the theory of potential. Other approaches include the First or Second kind Fredholm Integral Equations. One of the reasons that direct BIE are sometimes preferred is because the unknown function has immediate physical significance as compared to the unknown in the indirect BIE.

The behavior of water has been extensively researched on in various fields of engineering, physics and applied mathematics and much light has been shed on this aspect especially for simpler problems. The study of gravity free surface flows presents difficulties on the nonlinearity of the dynamic boundary condition in the free surface and also the fact that the location of this surface is not known *a priori*. This phenomenon has previously been investigated by physical models but numerical methods have now allowed more successful use of mathematical models to simulate this type of flow. In this thesis, free surface flow problems are investigated. They are highly nonlinear and must be treated with a combination of analytical and numerical methods of solution. In particular, free surface flows due to a point source are investigated. A boundary integral equation is formulated based on the Laplace equation and the application of the fundamental solution and Green’s identities.

FREE SURFACE FLOW PROBLEMS

Flows with moving boundaries are conspicuous all around us and include water waves, river flows, storm surges and tsunamis. Such flows are known as free surface flows. The term free comes from the fact that the surface is ‘free’ to move albeit only in ways permitted by the dynamics. The free surface is characterized by two conditions; the kinematic and dynamic free surface conditions (Faltisen, 1993). The pressure at the free surface is assumed to be constant and equal to the atmospheric pressure (Alexander Smits, 2014) and a particle on the free surface remains on the free surface, conditions referred to as dynamic free surface condition and kinematic free surface condition respectively. If there are no strong disturbances and the flow is steady, Bernoulli’s equation can be used along the surface. The shape of the free surface is a crucial part of the flow dynamics and free surface flows can often be well modeled by taking the pressure at the free surface to be a constant. Since the actual form of the free surface boundary is unknown, determining its shape is part of the solution. Free surfaces obey the Bernoulli’s principle yielding a nonlinear boundary condition along the free surface. Numerous methods which have often involved various approximations and numerical schemes have been employed previously to simplify free surface problems. The earliest works used the linearized theory obtained by neglecting the nonlinear terms in the free surface boundary conditions. Nowadays, the fully nonlinear problem can be solved using an accurate numerical method. Several numerical procedures can be used for solving free surface flows especially to describe with accuracy the motion of the free surface (Caboussat A., 2003). Hunter and Vanden-Broeck (1983) used a boundary integral equation method to construct unforced wave problem.

Accurate fully nonlinear solutions were obtained numerically by Vanden-Broeck and Dias (1992) for the forced problem in water of infinite depth. Other cases where numerical investigations have been undertaken include (Von Kerczek and Salvesen, 1977; Forbes and Schwartz, 1982; Binder et. al., 2006; 2007; King and Bloor, 1987; 1989; 1990). The methods have often involved various approximations and numerical schemes. The numerical simulation of interface flows can be a great ally in the understanding and improvement of free surface flows. Ideally, one wishes to solve the problem analytically as far as possible only to use numerical schemes later on in the solution process hence producing more accurate results. This basically leads to some basic assumptions and in this case the flow is assumed to be irrotational and the fluid is incompressible, meaning the density remains constant and inviscid, meaning it has no viscosity, and this leads to the system obeying Laplace's equation everywhere within the domain except in the neighborhood of singularities. The Laplace's equation occurs in the formulation of problems in many diverse fields of studies in engineering and physical sciences such as ideal fluid flow and flow in porous media. It will allow the problem to be treated analytically with boundary integral methods while numerical methods will be employed to solve the resultant integral equations.

SOURCE FLOW

Consider a flow where all streamlines are straight lines diverging from a central point O. The flow is referred to as a source flow. A point source in two-dimensional flow is a point where fluid enters and flows radially outward with a uniform angular distribution in all directions (Alexander Smits, 2014) as shown in Fig. 1. It is an example of a critical or singular point where streamlines can meet. Various authors have done works on flows due to a line source or point sink. Solutions with a stagnation point immediately above the source or sink have been given by Havelock (1926). Collings (1986) considered the flow due to a line source or sink within a fluid of finite depth and above a horizontal bottom, but with no restoring force, and found cusped solutions when the source/ sink was on the flat bottom and when the source depth was 0.56742 of the far fluid depth. King and Bloor (1988) used a conformal transformation and integral equation techniques to construct solutions to the steady flow induced by a submerged source beneath a cusped free surface and above a flat horizontal bottom where there is no restoring force and found explicit closed-form results for the equation of the free surface and the cusp height confirming the numerical and asymptotic results of Collings and Hocking. A subsequent paper by King and Bloor (1989) investigating the free surface flow of a uniform stream of ideal fluid around a Rankine body formed by a source and a sink where linear and nonlinear numerical solutions are presented in their work for a variety of body shapes for both supercritical and subcritical flows. Other authors include Tyvand (1992), Miloh and Tyvand (1993), Xue and Yue (1998) and Stokes et al (2003).

The mass flow supplied by the source is constant and since we have assumed the flow to be incompressible, the volume flow rate is also constant. The resulting velocity field for these flows therefore only has a radial component $\mathbf{u}_r$ which is inversely proportional to the distance from O. Due to the radial geometry, we shall use cylindrical polar coordinates to analyze source flows. At any radius, r, from the source, since the flow is purely radial, the tangential velocity is zero; and the radial velocity is the volume flow rate per unit depth, M, divided by the flow area per unit depth $2\pi r$.

Therefore,

$$\begin{aligned} \mathbf{u}_r &= \frac{M}{2\pi r} \\ \mathbf{u}_\theta &= \frac{1}{r}\frac{\partial u}{\partial \theta} = 0 \end{aligned}, \tag{1}$$

With these conditions in place, the potential and stream functions for a source are given as (Anderson, 2007):

$$\phi = \frac{M}{2\pi}\ln r \tag{2}$$

$$\psi = \frac{M}{2\pi}\theta \tag{3}$$

where M is the source strength, which is the rate of volume flow from the source per unit length; r is the distance from O and θ is in radian measure and taken in the range $0 \le \theta \le 2\pi$ and is measured positive in the anticlockwise direction.

The equipotential lines ($\phi =$ constant) are therefore concentric circles centered at the origin while the stream lines ($\psi =$ constant) are radial lines (See fig. 1). At the origin, r = 0, the velocity becomes infinite hence the line at the origin is a singularity in the flow field. The velocity potential and stream function are not continuous at a singular point and the circulation along a path through the origin cannot then be evaluated.

However, the circulation is zero for a path surrounding the origin since $\mathbf{u}_\theta = 0$ and the flow is irrotational everywhere but the origin. If either ϕ or ψ can be found, the velocity is known and the pressure can be obtained from Bernoulli's equation.

For a 2-D plane flow,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

is satisfied at every point except at the origin where the divergence is infinite. The origin is thus considered a singular point. A diagram of a source flow is shown in figure 1.

With the assumption that the fluid is inviscid and incompressible and the flow is steady, 2-D and irrotational, the stream function and the velocity potential are therefore harmonic functions of the x-y coordinates of the physical plane defined by the Cauchy-Riemann relations

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$
$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{5}$$

where u and v are velocity components. Then

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{-\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0 \tag{6}$$

The equation shows that if velocity is expressed in terms of the stream function, conservation of mass will be satisfied. The stream function is thus said to be a consequence of conservation of mass and is restricted to 2-D flows only.

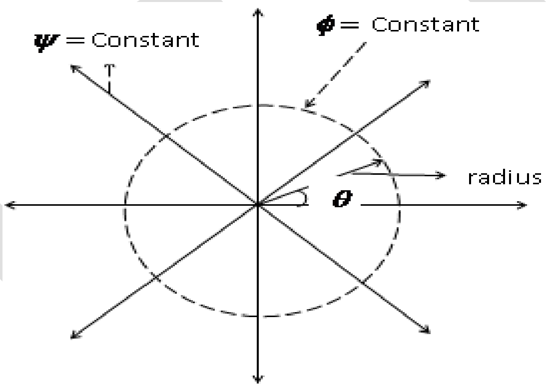

Figure 1: A Source

MATHEMATICAL FORMULATION

The motion of the fluid is assumed to be irrotational. Simple fluid dynamic theory shows that if a fluid is initially irrotational such as one would be if it were at rest and if there is no viscosity, then the fluid remains irrotational for the rest of its evolution. In this case, there exists a velocity potential ϕ such that the velocity vector $\mathbf{u}$ is given by the gradient $\mathbf{u} = \nabla \phi$ (Boas, 1983) which identically satisfies the condition for irrotationality, $\nabla \times \mathbf{u} = 0$.

Assuming also that the fluid is incompressible implying that mass conservation is satisfied by the equation $\nabla . \mathbf{u} = 0$ and substituting the condition for irrotationality, we obtain in the fluid domain $\Omega(t)$ with boundary $\partial\Omega(t)$

$$\nabla^2 \phi = 0 \qquad in \quad \Omega(t) \tag{7}$$

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It is well established that working with scalar quantities such as ϕ instead of vector quantities markedly simplifies the solution method (Thomson, 1974) as it reduces the problem from having three unknowns (u, v, w) to only one unknown ϕ hence this approach will be applied here. The velocity potential thus must satisfy Laplace's equation throughout the flow domain. In Cartesian coordinates, this becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad in \quad \Omega(t) \tag{8}$$

The boundary conditions of the flow problem must be introduced so that it can be solved. Solutions to equations such as this which are elliptic in nature do not possess real characteristics. The potential at any point depends continuously on the value of or its derivative normal to the boundary at all points around the boundary. An interior boundary value problem requires solving the Laplace's equation in the two-dimensional region bounded by a closed curve C subject to the boundary conditions:

$$\begin{aligned} \phi &= f(x, y) \quad for \quad (x, y) \in C_1 \\ \frac{\partial \phi}{\partial n} &= g(x, y) \quad for \quad (x, y) \in C_2 \end{aligned} \tag{9}$$

where f and g are suitably prescribed functions and C_1 and C_2 are non-intersecting curves such that $C_1 \cap C_2 = \{\ \}$ and $C_1 \cup C_2 = C$. The normal derivative $\partial \phi / \partial n$ in Eq. 9 is defined by:

$$\frac{\partial \phi}{\partial n} = n_x \frac{\partial \phi}{\partial x} + n_y \frac{\partial \phi}{\partial y} \tag{10}$$

where n_x and n_y are respectively the x and y components of a unit normal vector to the curve C.

Dynamic free surface condition

The dynamic boundary condition, for an inviscid fluid, refers to the pressure continuity condition through the free surfaces (Chan and Street, 1970; Nichols and Hirt, 1971) which is obtained from the momentum equation at the free surfaces. The dynamic boundary condition for free surface flows under gravity can be written as:

$$\begin{aligned} &\frac{\partial \phi}{\partial t} = -g\eta - \frac{1}{2}|\nabla \phi|^2 - \frac{P_0}{\rho} \\ &on \quad y = \eta(x, t) \end{aligned} \tag{11}$$

where P_0 is the pressure at the free surface, g the gravitational acceleration, η the vertical coordinate and ρ is the fluid density. The equation (11) is referred as the Bernoulli's Equation for unsteady, irrotational flow and applies throughout the fluid not just on a streamline. It is best viewed as an equation for the pressure. Solving a potential flow problem then becomes a two- step problem. First solve Laplace's equation for and then use Bernoulli's equation to compute for pressure.

Kinematic free surface condition

The kinematic boundary condition states that fluid particles, once at the free surface, must remain at the free surface. This therefore implies that the free surfaces move with the velocity of the fluid.

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= -\nabla\phi.\nabla\eta + \frac{\partial \phi}{\partial y} \\ on \quad y &= \eta(x,t) \end{aligned} \tag{12}$$

FREESPACE GREEN'S FUNCTION

It is also referred in many texts as the fundamental solution. It is the weighting function that is used in the boundary element formulation for a particular equation. The mathematical theory required to determine the fundamental solution of a constant coefficient PDE is well developed and has been used successfully to determine the fundamental solutions for a wide range of constant coefficient equations (Brebbia & Walker, 1980; Clements & Rizzo, 1978; Ortner, 1987). Fundamental solutions are known and have been published for the most important equations in engineering such as Laplace's equation, the diffusion equation and the wave equation (Brebbia, Telles & Wrobel, 1984). This does not however mean that it is guaranteed that the fundamental solution to a specific differential equation is known. The fundamental solution of any PDE is the analytical solution of the governing PDE under action of a point source and on an infinite domain. It satisfies the equation

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \delta(\xi - x, \eta - y) = 0 \tag{13}$$

On the domain $-\infty < x < \infty,\ -\infty < y < \infty.$ δ is the dirac delta function in 2-D, that is a point source of infinite strength at (ξ,η)

Using relations (1) and introducing the polar coordinates (r,θ) on a 2-D irrotational flow where the radius r is measured from the point (ξ,η), the plane polar components of the velocity are given by:

$$\begin{aligned} \nabla^2\phi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \\ &= 0 \end{aligned} \tag{14}$$

and

$$r = \sqrt{(\xi - x)^2 + (\eta - y)^2} \tag{15}$$

The δ term has disappeared from the equation because it is only nonzero when r = 0 and hence the delta function now acts as a boundary condition that states that $\phi \to \infty$ as $r \to 0$. The term $\frac{\partial^2 \phi}{\partial \theta^2}$ is also zero owing to symmetry and also because the domain is infinite.

The final equation is therefore:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left[\phi(r)\right]\right) = 0 \qquad \text{for } r \neq 0 \tag{16}$$

Integrating (16) twice gives:

$$\phi(r) = A\ln r + B \tag{17}$$

This solution satisfies the boundary condition at r = 0 since $\ln r \to \infty$ as $r \to 0$. The task is now to find the arbitrary constants A and B. In order to determine A, one has to take into account the magnitude of the source. Integrating (13) over a small area Ω_ε , of radius ε and center at $(x, y) = (\xi, \eta)$ one obtains:

$$\begin{aligned} -\int_{\Omega_\varepsilon} \nabla^2 \phi d\Omega_\varepsilon &= \int_{\Omega} \delta(\xi - x, \eta - y) d\Omega \\ &= 1 \end{aligned} \tag{18}$$

from the selection property of the dirac delta function which means that when involved in an integration process with another function, it selects the value of the other function at the point where the $\delta - function\ \ \delta(\xi - \mathbf{x})$ has a zero argument at $\xi = x$

This means that $\int_{\Omega} \nabla^2 \phi d\Omega = -1$.

We also assume that Ω is a circle of radius $\varepsilon > 0$ centered at r = 0. By Green-Gauss theorem,

$$\int_{\Omega} \nabla^2 \phi d\Omega = \int_{\partial\Omega} \frac{\partial \phi}{\partial n} ds \tag{19}$$

where n is the normal vector and s is the distance along the path counter clockwise. Since r and n always point in the same direction when Ω is a circle, this is equivalent to

$$\int_{\partial\Omega} \frac{\partial \phi}{\partial r} ds \tag{20}$$

Differentiating (13), we have

$$\frac{\partial \phi}{\partial r} = \frac{A}{r} = \frac{A}{\varepsilon} \tag{21}$$

We therefore rewrite (20) and evaluate to have:

$$\int_0^{2\pi\varepsilon} \frac{A}{\varepsilon} ds = \frac{A(2\pi\varepsilon)}{\varepsilon} = -1 \tag{22}$$

This implies that

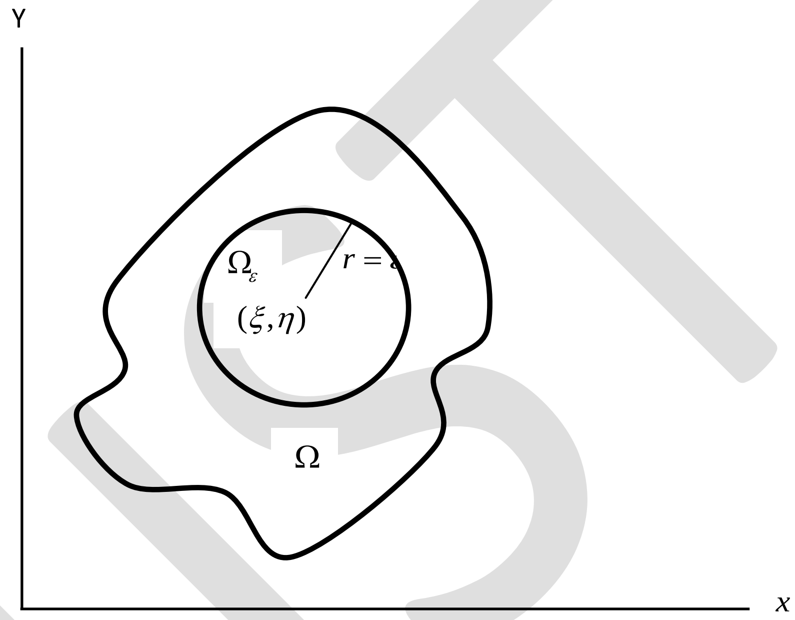

Figure 2: Domain used to evaluate Fundamental Solution

$$A = \frac{-1}{2\pi} \tag{23}$$

Since B is an arbitrary constant, letting it to be zero, the fundamental solution for Laplace's equation also referred as the free-space Green's function is thus:

$$\phi = \frac{-1}{2\pi} \ln r \tag{24}$$

Denoting (24) by $G(\xi, \eta; x, y)$, we write:

$$G(\xi, \eta; x, y) = \frac{-1}{4\pi} \ln\left[(\xi - x)^2 + (\eta - y)^2\right] \tag{25}$$

which is the fundamental solution of the two dimensional Laplace's equation and satisfies equation (8) everywhere except at $(\xi,\eta) \in \Omega$ where it is not well defined. The availability of a fundamental solution is an essential requirement for the construction of a boundary integral equation method. The fundamental solution of the time-dependent problem always has a representation by a Laplace integral of the frequency-dependent fundamental solution of the corresponding elliptic problem.

NUMERICAL MATHEMATICAL MODEL

The boundary integral equation (BIE) formulation and the numerical solutions using boundary element methods (BEM) for mechanics problems originated about 50 years ago. The 2-D potential flow problem was first formulated in terms of a direct BIE and solved by Jawson (Jawson, 1963). This work was later extended by Rizzo to the vector case – 2-D elastostatic problem (Rizzo, 1967). Following these early works, extensive research efforts have been made for the development of BIE/BEM. Some important textbooks and research items can be found in (Brebbia, 1978; Banerjee, 1979 – 1991; Brebbia and Domininguez, 1989; Kane, 1994, Cheng and Cheng, 2005). Integral equations afford a powerful means of attacking the boundary value problems of potential theory and classical elasticity (Jawson, 1963). Boundary methods are known for their major advantage to restrict the discretization only to the boundary of a body. The advantage of this is that the dimension of the problem is effectively reduced by one. The BIM transforms the given PDE into an equivalent set of integral equations over the boundary of the solution domain. The numerical solution of the boundary integral equations only requires subdividing the boundary curve of the solution domain. A Green's identity and the fundamental solution of the PDE are used to transform the original PDE problem into a problem which involves only boundary integrals (Pomeranz, 2008). The most reputed method is the boundary integral equation method otherwise known as Boundary Element Method (BEM), a name resulting from the technique used to solve boundary integral equations (Katsikadelis, 2002).

Together with the fundamental solution (25) the Laplace equation (7) can be used to derive a useful boundary integral solution. Multiplying the Laplace equation by the fundamental solution and integrating over the domain to find a weak solution we have:

$$G\left[\nabla^2\phi\right]=0 \quad \Rightarrow \quad \int_\Omega \left[\nabla^2\phi G\right] d\Omega = 0 \tag{26}$$

The fundamental equation acts as a weighting function. Using the Green – Gauss theorem gives:

$$\int_\Omega \left[\nabla^2\phi G\right] d\Omega = \int_{\partial\Omega} \frac{\partial\phi}{\partial n} G ds - \int_\Omega \nabla\phi\nabla G d\Omega \tag{27}$$

where $\partial\Omega$ is the boundary of domain Ω and is assumed to be smooth enough so that integration by parts formula holds. Integrating by parts in order to get a second derivative on the weighting function leads to:

$$\int_{\Omega} \left[\nabla^2 \phi G \right] d\Omega \quad = \quad \int_{\partial\Omega} \frac{\partial \phi}{\partial n} G ds - \int_{\partial\Omega} \frac{\partial G}{\partial n} n ds + \int_{\Omega} \phi \nabla^2 G d\Omega \tag{28}$$

where $\frac{\partial \phi}{\partial n}$ and $\frac{\partial G}{\partial n}$ are the outward normal derivatives of ϕ and G respectively on the boundary of domain Ω .

The last term in (28) can be written as:

$$\begin{aligned} \int_{\Omega} \phi \nabla^2 G d\Omega \quad &= \quad -\int_{\Omega} \phi \delta(\xi - x, \eta - y) d\Omega \\ &= \quad -\phi(\xi,\eta) \quad (\xi,\eta) \quad \in \quad \Omega \end{aligned} \tag{29}$$

that is, the domain integral has been replaced by a point value (Hunter and Pullan, 2001). This gives the boundary integral equation:

$$\begin{aligned} -\int_{\partial\Omega} \frac{\partial \phi}{\partial n} G ds + \int_{\partial\Omega} \frac{\partial G}{\partial n} n ds + \phi(\xi,\eta) \quad &= \quad 0 \\ for \quad (\xi,\eta) \quad \in \Omega & \end{aligned} \tag{30}$$

It is therefore possible from the foregoing to find ϕ at an arbitrary point $(\xi,\eta) \in \Omega$ by looking at ϕ and G only on the boundary. But for this to be possible, we first find ϕ and/or $\frac{\partial \phi}{\partial n}$ on the boundary.

Suppose $(\xi,\eta) \notin \Omega$,

$$\begin{aligned} \int_{\Omega} \phi \nabla^2 G d\Omega \quad &= \quad -\int_{\Omega} \delta(\xi - x, \eta - y) d\Omega \\ &= \quad 0 \end{aligned} \tag{31}$$

This is so because δ is zero everywhere in Ω except at (ξ,η) where it is infinite.

Assume that some point is on $P(\xi,\eta)$ is on $\partial\Omega$. Define a disk with radius ε around P (as shown in Figure 3).

The point P is often called the source point and (x, y) the field point, or point at which we wish to measure the effect of the source a distance r away from it. Both points P and (x, y) can be inside the domain, but when applying BEM, one point or the other becomes a boundary point. The fundamental solution thus has a singularity at (x,y) since r = 0. We now have two subdomains on the boundary $\partial\Omega = \partial\Omega_{-\varepsilon} + \partial\Omega_{\varepsilon}$. Taking the limit of each domain as $\varepsilon \to 0$, we now look at the following integrals:

$$\begin{aligned}
\int_{\partial\Omega_\varepsilon} \phi \frac{\partial G}{\partial n} ds &= \int_{\partial\Omega_\varepsilon} \phi \frac{\partial}{\partial n}\left(\frac{-1}{2\pi}\ln r\right) \\
&= \int_{\partial\Omega_\varepsilon} \frac{-\phi}{2\pi r} ds \\
&= \frac{-1}{2\pi r}\int_{\partial\Omega_\varepsilon} \phi ds \qquad (32)\\
&= -\phi(P)\frac{\pi\varepsilon}{2\pi r} \\
&= \frac{-\phi(P)}{2}
\end{aligned}$$

where r = ε is fixed since the domain is a circle and in the limit of the integral is the value at P by the mean

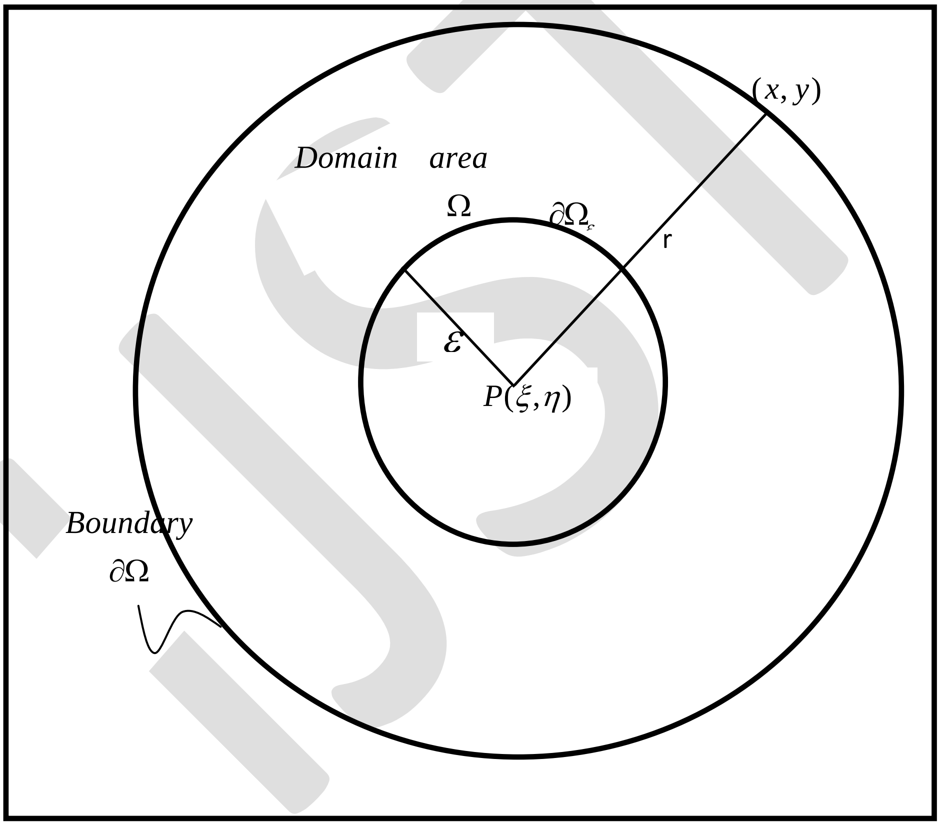

Figure 3: A Small disk $\partial\Omega_\varepsilon$ enclosing $P(\xi,\eta)$ within Ω

value theorem.

Similarly:

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} G\frac{\partial\phi}{\partial n} ds &= \frac{-\ln\varepsilon}{2\pi}\int_{\partial\Omega_\varepsilon}\frac{\partial\phi}{\partial n} ds \qquad (33)\\
&= \frac{-\ln\varepsilon}{2\pi}\frac{\partial\phi(P)}{\partial n}\pi\varepsilon \to 0
\end{aligned}$$

For the two integrals over, we have:

$$\lim_{\varepsilon \to 0} \int_{\partial\Omega_{-\varepsilon}} G \frac{\partial \phi}{\partial n} ds = \int_{\partial\Omega} G \frac{\partial \phi}{\partial n} ds \tag{34}$$

$$\lim_{\varepsilon \to 0} \int_{\partial\Omega_{-\varepsilon}} \phi \frac{\partial G}{\partial n} ds = \int_{\partial\Omega} \phi \frac{\partial G}{\partial n} ds \tag{35}$$

From (32) – (35) we have:

$$\phi(P) + \int_{\partial\Omega} \phi \frac{\partial G}{\partial n} ds = \frac{1}{2}\phi(P) + \int_{\partial\Omega} G \frac{\partial \phi}{\partial n} ds \tag{36}$$

$$\frac{1}{2}\phi(P) + \int_{\partial\Omega} \phi \frac{\partial G}{\partial n} ds = \int_{\partial\Omega} G \frac{\partial \phi}{\partial n} ds \tag{37}$$

for $\phi(P)$ on $\partial\Omega$ and $\partial\Omega$ smooth at P. If $\partial\Omega$ is not smooth at P, then:

$$\left(1 - \frac{\alpha}{2\pi}\right)\phi(P) + \int_{\partial\Omega} \phi \frac{\partial G}{\partial n} ds = \int_{\partial\Omega} G \frac{\partial \phi}{\partial n} ds \tag{38}$$

where α is the interior angle of the corner at P.

In general,

$$\lambda(P)\phi(P) + \int_{\partial\Omega} \phi \frac{\partial G}{\partial n} ds = \int_{\partial\Omega} G \frac{\partial \phi}{\partial n} ds \tag{39}$$

where the integral free terms $\lambda(P)$ are defined as:

$$\lambda(P) = \begin{cases} 1 & P \in \Omega \\ \frac{1}{2} & P \in \partial\Omega, \quad \partial\Omega \quad Smooth \\ 1 - \frac{\alpha}{2\pi} & P \in \partial\Omega, \quad \partial\Omega \quad Notsmooth \\ 0 & P \notin \Omega \end{cases} \tag{40}$$

It can be shown (Debanath and Mikusinsky, 2005) that the above result can be generalized to the case of higher dimensions than 2 for continuous functions ϕ in $\mathbb{R}^m$ with compact support (i.e. the closure of the set of points $x \in \mathbb{R}^m$ outside where ϕ vanishes). This form of equation has been used by different researchers such as

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(Katsikadelis, 2002). Since the solution relies on the boundary points, we let $\lambda(P) = \frac{1}{2}$ and therefore the boundary integral equation will be:

$$\frac{1}{2}\phi(\bar{\xi}) = \int_{\partial\Omega} G(\bar{\xi};\bar{x})\frac{\partial\phi(\bar{x})}{\partial n}ds - \int_{\partial\Omega} \phi(\bar{x})\frac{\partial G(\bar{\xi};\bar{x})}{\partial n}ds \qquad (\bar{\xi}) \in \Omega \tag{41}$$

where $\bar{\xi} = (\xi, \eta)$ and $\bar{x} = (x, y)$

$$G(\bar{\xi}, \bar{x}) = \frac{-1}{2\pi}\ln r \tag{42}$$

$$\frac{\partial G(\bar{\xi}, \bar{x})}{\partial n} = \frac{1}{2\pi r}\frac{\partial r}{\partial n} \tag{43}$$

with r being the distance between the collocation point $\bar{\xi}$ and the field point $\bar{x}$

Equation (39) involves only the surface distributions of ϕ and $\frac{\partial\phi}{\partial n}$, and the value of ϕ at point P. Once the surface distributions of ϕ and $\frac{\partial\phi}{\partial n}$ are known, the value of ϕ at any point P inside Ω can be found since all surface integrals in (39) are then known. The procedure is thus to use (39) to find the surface distributions of ϕ and $\frac{\partial\phi}{\partial n}$. Since an integral equation is difficult to solve, the boundary integral equations are implemented discretely on elements. Due to this discrete implementation, the boundary integral equation method is commonly known as boundary element method (BEM).

SUMMARY

BEM requires a fundamental solution for the governing equation and is therefore mainly applicable for linear differential equations. It is a numerical method that discretizes the solution boundary only and it is possible therefore to obtain better accuracy with a fewer number of elements. By requiring the discretization of the boundary only rather than the domain, the BEM has a much smaller system of equations and this results in a considerable reduction of data required to run a program. As a consequence of the boundary discretization, the BEM is a suitable method for problems on external domains or domains that have a free or moving boundary.

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