Coupled Algorithms Laplace Iterative Method with Fractional Lagrange Multiplier for Solving a System of Space-Time Fractional Order Differential-Algebraic Equations

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Abstract: This work provided the involution of the algorithm for analytic solution of space- and time- fractional differential algebraic equations. The algorithm referred to good effective method for combination the Laplace Iterative method with complex Lagrange multiplier. The fractional derivative is described in Caputo sense. Illustrative example are included to demonstrate the validity and applicability of the presented technique.

Keywords: Caputo Derivative; Analytic solution; Laplace Iteration Method; Mittag-Leffller Functions; Space-time Fractional order differential-algebraic system.

1 Introduction

Differential equations of fractional order have gained much attention due to the tremendous use in fluid mechanics, mathematical biology, electrochemistry, physics, and so on [1- 4]. Recently, many important mathematical models can be expressed in terms of system of differential- algebraic equations of fractional order. The solution of fractional differential equations is so much difficult [5-6]. In general, there exists no method that yields an exact solution for fractional differential –algebraic equations. Only approximate solution can be derived by using linearization or perturbation methods.

In recent years, many researchers have focused on the numerical solution of fractional differential –algebraic equations. Some numerical methods have been developed, such as implicit Runge- Kutta method [7], Padé approximation method [8-11], homotopy perturbation method [12-16], Adomian decomposition method [17-21], homotopy analysis method [22-23], variation iteration method [24-26], homotopy analysis transform method [27].

In 2013, Habibolla et al. [28], presented an alternative approach based on Laplace iterative method (LIM) for finding series solutions to linear and nonlinear systems of PDEs. The applied method gave rapidly convergent successive approximations.

The objective of the present paper is making combination of the Laplace Iteration method with the Lagrange multiplier (LLIM) to provide approximate solution for linear and nonlinear space-time fractional order of differential-algebraic equations. The efficient and accuracy of the method used in this paper will be demonstrated by comparison with the known exact solutions in the non-fractional case. The fractional derivatives are described in the Caputo sense.

The structure of the paper described as follows. In section 2, we give the concept of fractional calculus. In section 3, we now identify the complex fractional Lagrange multipliers. In section 4, we give a brief description of how the method works and propose an algorithm with Lagrange multiplier. LLIM for system of space-time fractional order of differential-algebraic equations. In section 5. we consider some illustrative examples. Finally, in section 6, we have presented our conclusions.

2. Preliminaries and notations

In this section, we give some basic definitions and properties of fractional calculus theory which will be used in this paper.

Definition 2.1 [29] A real function f(t), t >0, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty]$ and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2.2 The Riemann-Liouville fractional order integral of $\alpha > 0$ of function $f(t) \in C\mu, \mu > -1$ is defined as [29] :

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \qquad \ldots(2.1)$$

$$J^0 f(t) = f(t)$$

Definition 2.3 The fractional derivative of function $f(t)$ in Caputo sense is defined as [29]:

$$D_*^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \quad t > 0 \qquad \ldots(2.2)$$

For $m-1 < \alpha \le m,\ m \in N,\ t > 0,\ f \in C_{-1}^m$.

Definition 2.4 [30] The single parameter and the two parameters variants of the Mittag- Leffler function are denoted by $E_\propto(t)$ and $E_{\propto,\ \beta}(t)$, respectively, which are relevant for their connection with fractional calculus, and are defined as:

$$E_\propto(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\propto k+1)}, \quad \propto > 0, t \in C \qquad \ldots(2.3)$$

$$E_{\propto,\ \beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\propto k+\beta)}, \quad \propto,\ \beta > 0, t \in C \qquad \ldots(2.4)$$

For special choices of the values of the parameter $\propto,\ \beta$ we obtain well-known classical functions, e.g.:

$$E_1(t) = e^t, \quad E_{\propto,\ 1}(t) = E_\propto(t).$$

As we will see later, classical derivatives of the Mittag-Leffler function appear in solution of FDEs. Since the series (2.4) is uniformly convergent we may differentiate term by term and obtain

$$E^m{}_{\propto,\ \beta}(t) = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{t^k}{\Gamma(\propto k+\alpha m+\beta)} \qquad \ldots(2.5)$$

Definition 2.5 The Laplace transform $\mathcal{L}[x(t)]$ of the Caputo fractional derivative is given as [29]:

$$\mathcal{L}[D_*^\alpha f(t)] = s^\alpha \mathcal{L}[f(t)] - \sum_{k=0}^{m-1} s^{(\alpha-k-1)} f^{(k)}(0), \quad m-1 < \alpha \le m. \qquad \ldots(2.6)$$

Lemma 2.6 [30] For $\alpha \ge \beta,\ \alpha > \gamma,\ a \in R,\ s^{\alpha-\beta} > |a|$ and $|s^\alpha + a\,s^\beta| > |b|$ we have

$$\mathcal{L}^{-1}\left(\frac{s^{\alpha-\beta}}{s^\alpha + a\,s^\beta + b}\right) = x^{\alpha-\gamma-1} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-b)^n (-a)^k \binom{n+k}{k}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma)} x^{k(\alpha-\beta)+n\alpha} \qquad \ldots(2.7)$$

3. Basics of the Fractional Variational Iteration Method

We consider a general nonlinear fractional differential equation:

$$L_\alpha \mathrm{x(t)} + R_\alpha \mathrm{x(t)} + N_\alpha \mathrm{x(t)} = \mathrm{f(t)},\ x \in R,\ 0 < \alpha \le 1 \qquad \ldots(3.1)$$

Where L_α denotes linear fractional derivative operator of order 2α, R_α denotes linear fractional derivative operator of order less than L_α, N_α denotes nonlinear fractional operator, and f(t) is the nondifferentialle source term.

According to the rule of fractional variational iteration method, the correction fractional functional for (3.1) is constructed as Hossein and Hassan [31]:

$$x_{n+1}(t) = x_n(t) + {}_0I_t^\propto\left[\frac{\lambda(s)^\alpha}{\Gamma(1+\alpha)}\left(L_\alpha \mathrm{x(s)} + R_\alpha \tilde{x}_n(s) + N_\alpha \tilde{x}_n(s) - \mathrm{f(s)}\right)\right] \qquad \ldots(3.2)$$

Where ${}_0I_t^\propto$ is the Riemann- Liouville fractional integral operator of order α, $\lambda(s)$ is called the general fractional Lagrange multiplier which can be identified optimally via variational theory, x_n is the nth approximate solution. Making the fractional variation of (3.2), we have

$$\delta^\alpha x_{n+1}(t) = \delta^\alpha x_n(t) + {}_0I_t^\propto \delta^\alpha[\lambda(s)(L_\alpha \mathrm{x}_n(s) + R_\alpha \tilde{x}_n(s) + N_\alpha \tilde{x}_n(s) - \mathrm{f(s)})] \qquad \ldots(3.3)$$

The extremum condition of x_{n+1} is given by [31],

$$\delta^{\alpha} x_{n+1}(t) = 0 \qquad \ldots (3.4)$$

In view of (3.4), we have the following stationary conditions:

$$1 - \left(\lambda(s)\right)^{(\alpha)}\Big|_{s=t} = 0, \quad \lambda(s)|_{s=t} = 0, \quad \left(\lambda(s)\right)^{(2\alpha)}\Big|_{s=t} = 0 \qquad \ldots(3.5)$$

So, from (3.5), we get $\lambda(s) = \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)}$.

4. Description of the new method (LLIM)

In this article, we consider the following non-homogenous, non-linear system of fractional order differential-algebraic equations

$$\left.\begin{aligned} D_*^{2\propto_i} x_i(t) + A D_*^{\propto_i} x_i(t) + B x_i(t) &= h_i(t, x_1, x_2, \ldots, x_n) \\ 0 &= g(t, x_1, x_2, \ldots, x_n) \end{aligned}\right\} \qquad \ldots (4.1)$$

With initial conditions $x_i(0) = a_i, \acute{x}_i(0) = b_i \quad i = 1, 2, \ldots, n-1, \; x_n(0) = c_i$ Here $D_*^{\propto_i}$ is Caputo fractional derivative of order $\propto_i$, satisfying the relation $0 < \propto_i \leq 1$. Eq. (4.1) can be rewritten as:

$$\left.\begin{aligned} L_1 x_1(t) + N_1(t, x_1, x_2, \ldots, x_n) &= f_1(t) \\ L_2 x_2(t) + N_2(t, x_1, x_2, \ldots, x_n) &= f_2(t) \\ &\;\;\vdots \\ L_{n-1} x_{n-1}(t) + N_n(t, x_1, x_2, \ldots, x_n) &= f_{n-1}(t) \\ 0 &= g(t, x_1, x_2, \ldots, x_n) \end{aligned}\right\} \qquad \ldots(4.2)$$

Where L_i is a linear operator, N_i a nonlinear operator and $f_i(t)$ is an inhomogenous item form i=1, 2, …, n-1. Eq. (4.2) can be rewritten down as a correction function in the following way:

$$L_i x_i(t) = f_i(t) - N_i(t, x_1, x_2, \ldots, x_n) = R_i(t, x_1, x_2, \ldots, x_n), \quad i=1, 2, \ldots, n-1 \qquad \ldots(4.3)$$

Therefore:

$$L_i x_i(t) = R_i(t, x_1, x_2, \ldots, x_n), \quad i=1, 2, \ldots, n-1.$$

The Laplace Iteration Method assumed a series solution for x_i given by an infinite sum of components:

$$x_i(t) = \lim_{n\to\infty} x_i^p(t) = \lim_{n\to\infty} \sum_{j=0}^{p} v_i^j(t), \quad i = 1, 2, \ldots, n-1 \qquad \ldots(4.4)$$

In which x_i^n indicates the n-th approximation of x_i, where v_i^j is the j^{th} component of the solution of x_i and v_i^0 is the solution of $L_i x_i = 0$ along with the following initial conditions of the main problem:

$$v_i^1(t) = \varphi_i(v_i^0)$$

$$v_i^{k+1}(t) = \varphi_i\left(\sum_{j=0}^{k} v_i^j(t)\right) - \sum_{j=0}^{k} v_i^j(t), \qquad k \geq 1$$

In which $\varphi_i(v_i^k)$ is obtained as follows:

$$L_i \varphi_i(v_1^k, v_2^k, \ldots, v_n^k) = R_i(t, x_1, x_2, \ldots, x_n), \qquad \ldots(4.5)$$

Using the homogenous initial conditions, supposing that L_i linear operator, therefore, taking Laplace transform to both sides of Eq. (4.5) in the usual way and using the homogenous initial conditions, the result can be obtained as following:

$$p_i(s).\Phi_i^k(s) = \mathcal{R}_i(v_i^k(s)), \qquad \ldots(4.6)$$

Where $\mathcal{L}[\varphi_i(v_1^k, v_2^k, \ldots, v_n^k)] = \Phi_i^k$, $p_i(s)$ is a fractional polynomial with the fractional degree of the highest derivative in Eq. (4.6) (The same as the highest order of the linear operator L_i). Thus,

$$\mathcal{L}[w] = \varpi, \; \psi_i(s) = \frac{1}{p_i(s)}, \; \mathcal{L}[u_i(t)] = \psi_i(s) \qquad \ldots(4.7)$$

In Equations (4.5) and (4.6), the function $\mathcal{R}_i(v_i^k(s))$ and $R_i(t, x_1, x_2, \ldots, x_n)$ are abbreviated as $\mathcal{R}_i$ and R_i respectively. Hence, Eq. (4.6) is rewritten as:

$$\Phi_i^k(s) = \mathcal{R}_i((v_1^k, v_2^k, \ldots, v_n^k)(s)) \,.\, \psi_i(s) \qquad \ldots(4.8)$$

Now, by applying the inverse Laplace Transform to both side of Eq. (4.8) and using the convolution Theorem, the following relation can be presented:

$$\emptyset_i(v_1^k, v_2^k, \ldots, v_n^k) = \int_0^t R_i((v_1^k, v_2^k, \ldots, v_n^k)(\tau)) \,.\, u_i(t-\tau)\, d\tau \qquad \ldots(4.9)$$

Therefore

$$x_i^{p+1}(t) = \sum_{j=0}^{p+1} v_i^j(t) = x_i^0(t) + \int_0^t R_i(x_i^p(\tau)) \,.\, u_i(t-\tau)\, d\tau, \; i = 1, 2, \ldots, n-1 \qquad \ldots(4.10)$$

After identifying the initial approximation of x_i^0, the remaining approximations $x_i^p, p > 0$ can be determined so that each term can be determined by previous terms and the approximation of iteration formula can be entirely evaluated.
Consequently, the exact solution may be obtained by:

$x_i = \lim_{p\to\infty} x_i^p(t) = \lim_{p\to\infty} \sum_{j=0}^{p} v_i^j(t)$, $i = 1, 2, \dots, n-1$. …(4.11)

Which is the Laplace Iteration method.
Now, we can construct a correct function as follows:

$x_{n+1}(t) = x_0(t) + \int_0^t \lambda(\tau) R_i((v_1^k, v_2^k, \dots, v_n^k)(\tau)) \,.\, u_i(t-\tau)\, \mathrm{d}\tau$ …(4.12)

Where $\lambda(\tau)$ fractional Lagrange multiplier.

5. Applications and Results

In this part, we introduce some applications on LLIM to solve differential- algebraic equations with time- space fractional derivatives

Example 1: consider the following system of nonlinear space-time fractional order differential – algebraic equations :

$$\left.\begin{aligned} D_*^{2\alpha}x(t) + D_*^{\alpha}x(t) - 2\,x(t) + x(t)z(t) = 1,\\ D_*^{2\alpha}z(t) + 2D_*^{\alpha}z(t) + z(t) - y(t) + x^2(t) = 0,\\ y(t) + x^2(t) = 0\,, \quad t \in [o,1],\ 0 < \alpha \le 1 \end{aligned}\right\} \quad \dots(5.1)$$

Subject to initial conditions x(0)=y(0)=z(0)=1and $\acute{x}(0) = 1$, $\acute{z}(0) = -1$. For the special case $\alpha = 1$, we have analytical solution $x(t) = e^t$, $z(t) = e^{-t}, and\ y(t) = e^{2t}$.

Solution:

From the Eq. (5.1), optimal selection auxiliary linear operator the equation is represented as follows:
$L_1x(t)$: $D_*^{2\alpha}x(t) + D_*^{\alpha}x(t) - 2\,x(t)$
$L_2z(t)$: $D_*^{2\alpha}z(t) + 2D_*^{\alpha}z(t) + z(t)$
Therefore $\emptyset_i(v_1^k, v_2^k, v_3^k)$, i=1,2; are defined as:

$$\left.\begin{aligned} \emptyset_1(v_1^k, v_2^k, v_3^k) &= \int_0^t u(t-\tau)[\,1 - v_2^k(\tau)v_1^k(\tau)]d\tau \\ \emptyset_2(v_1^k, v_2^k, v_3^k) &= \int_0^t u(t-\tau)[\,v_3^k(\tau) - (v_1^k(\tau))^2]d\tau \end{aligned}\right\} \quad \dots(5.2)$$

Then, using Eq. (5.2), the Laplace Iteration Method with Fractional Lagrange Multiplier formulae in t-direction for the calculation of the approximate solution of Eq.(5.3) can be readily obtained as:

$$\left.\begin{aligned} x_{n+1}(t) &= x_0(t) + \int_0^t [\lambda(\tau)u_1(t-\tau)(1 - z_n(\tau)x_n(\tau)]\mathrm{d}\tau,\\ z_{n+1}(t) &= z_0(t) + \int_0^t [\lambda(\tau)u_2(t-\tau)(y_{n+1}(\tau) - x_{n+1}^2(\tau)]\mathrm{d}\tau,\\ y_n &= x_n^2(t) \end{aligned}\right\} \quad \dots(5.3)$$

Case 1: $\alpha = 1$

$$L_1x(t): D_*^2x(t) + D_*^1x(t) - 2\,x(t)$$
$$\Rightarrow p_1(s) = s^2 + s - 2$$
$$\Rightarrow \psi_1(s) = \frac{1}{p_1(s)} = \frac{1}{s^2 + s - 2}$$
$$\Rightarrow u_1(t) = \mathcal{L}^{-1}[\psi_1(s)] = -\frac{1}{3}e^{2t} + \frac{1}{3}e^{t}$$

And

$$L_2z(t): D_*^2z(t) + 2D_*^1z(t) + z(t)$$
$$\Rightarrow p_2(s) = s^2 + 2s + 1$$
$$\Rightarrow \psi_2(s) = \frac{1}{p_2(s)} = \frac{1}{s^2 + 2s + 1}$$
$$\Rightarrow u_2(t) = \mathcal{L}^{-1}[\psi_2(s)] = te^{-t}$$

where the initial approximation must be satisfied by the following equations:

$$L_1x(t) = 0, \qquad x(0) = 1, \qquad \acute{x}(0) = 1 \Longrightarrow x_0(t) = v_1^0 = e^t$$
$$L_2z(t) = 0, \qquad z(0) = 1, \ \acute{z}(0) = -1 \Longrightarrow z_0(t) = v_2^0 = e^{-t}$$
$$y_0(t) = e^{2t}$$

Accordingly, by Eq. (5.3) the higher order approximation of the exact solution can be obtained as follows:

$$x_1(t) = \sum_{i=0}^{1} v_1^k(t) = e^t$$
$$z_1(t) = \sum_{i=0}^{1} v_2^k(t) = e^{-t}$$

The remaining approximations $x_n = 0, z_n = 0, n > 1$ can be completely determined such that each term will be determined using the prevous term: thus, the exact solution is as follows:

$$x(t) = \lim_{p\to\infty} \sum_{i=0}^{p} v_1^k(t) = e^{-t}$$
$$z(t) = \lim_{p\to\infty} \sum_{i=0}^{p} v_2^k(t) = e^{t}$$
$$y(t) = e^{2t}$$

Case 2: $\boldsymbol{\alpha = 0.6}$

$$L_1x(t) \colon D_*^{1.2}x(t) + D_*^{0.6}x(t) - 2\,x(t)$$
$$\Longrightarrow p_1(s) = s^{1.2} + s^{0.6} - 2$$
$$\Longrightarrow \psi_1(s) = \frac{1}{p_1(s)} = \frac{1}{s^{1.2} + s^{0..6} - 2}$$
$$\Longrightarrow u_1(t) = \mathcal{L}^{-1}[\psi_1(s)] = t^{0.2} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(2)^n\,(-1)^k\binom{n+k}{k}}{\Gamma\big(0.6\,k + 1.2(n+1)\big)} t^{0.6\,k+1.2n}$$

And

$$L_2z(t) \colon D_*^{1.2}z(t) + 2D_*^{0.6}z(t) + \ z(t)$$
$$\Longrightarrow p_2(s) = s^{1.2} + 2s^{0.6} + 1$$
$$\Longrightarrow \psi_2(s) = \frac{1}{p_2(s)} = \frac{1}{s^{1.2} + 2s^{0.6} + 1}$$
$$\Longrightarrow u_2(t) = \mathcal{L}^{-1}[\psi_2(s)] = t^{0.2} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^n\,(-2)^k\binom{n+k}{k}}{\Gamma\big(0.6\,k + 1.2(n+1)\big)} t^{0.6\,k+1.2n}$$

where the initial approximation must be satisfied by the following equations:

$$L_1x(t) = 0, \qquad x(0) = 1, \acute{x}(0) = 1 \Longrightarrow x_0(t) = v_1^0$$
$$= \sum_{m=0}^{\infty} \frac{2^m}{m!}\Bigg[t^{1.2\,m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.6})^k}{\Gamma(0.6k + 0.6m + 0.6m + 1)} + t^{1.2\,m+0.6} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.6})^k}{\Gamma(0.6k + 0.6m + 0.6m + 1.6)} + t^{1.2\,m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.6})^k}{\Gamma(0.6k + 0.6m + 0.6m + 2)}\Bigg]$$
$$L_1z(t) = 0, \qquad z(0) = 1, \acute{z}(0) = -1 \Longrightarrow z_0(t) = v_2^0$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left[t^{1.2m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+1)} + t^{1.2m+0.6} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+1.6)} + t^{1.2m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+2)}\right]$$

$y_0(t)$
$= \left(\sum_{m=0}^{\infty} \frac{2^m}{m!}\left[t^{1.2\,m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+1)} + t^{1.2\,m+0.6} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+1.6)} + \right.\right.$
t1.2 m+1k=0∞k+m!k!(−t0.6)kΓ 0.6k+0.6m+0.6m+22

Accordingly, by Eq. (5.3) the higher order approximation of the exact solution can be obtained as follows:

$$x_1(t) = \sum_{i=0}^{1} v_1^k(t) = \sum_{m=0}^{\infty} \frac{2^m}{m!}\left[t^{1.2\,m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+1)} + t^{1.2\,m+0.6} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+1.6)} + t^{1.2\,m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+2)}\right] + \int_0^t \left(\frac{(\tau-t)^{0.6}}{\Gamma(0.6+1)}\right)$$

$$\left((t-\tau)^{0.2} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(2)^n\,(-1)^k \binom{n+k}{k}}{\Gamma\big(0.6\,k+1.2(n+1)\big)} (t-\tau)^{0.6\,k+1.2n}\right)$$

$$\left(\begin{array}{l} 1 - \left(\sum_{m=0}^{\infty} \frac{2^m}{m!}\left[\begin{array}{l} t^{1.2m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+1)} \\ +t^{1.2m+0.6} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+1.6)} \\ +t^{1.2\,m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+2)} \end{array}\right]\right) \\ \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left[\begin{array}{l} t^{1.2m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+1)} \\ +t^{1.2m+0.6} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+1.6)} \\ -t^{1.2\,m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.6})^k}{\Gamma(0.6k+0.6m+0.6m+2)} \end{array}\right]\right) \end{array}\right) d\tau$$

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Case 1: $\alpha = 0.75$

$$L_1 x(t) \colon D_*^{1.5} x(t) + D_*^{07.5} x(t) - 2\,x(t)$$
$$\Rightarrow p_1(s) = s^{1.5} + s^{0.75} - 2$$
$$\Rightarrow \psi_1(s) = \frac{1}{p_1(s)} = \frac{1}{s^{1.5} + s^{0.75} - 2}$$
$$\Rightarrow u_1(t) = \mathcal{L}^{-1}[\psi_1(s)] = t^{0.5} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(2)^n\,(-1)^k \binom{n+k}{k}}{\Gamma\big(0.75\,k+1.5(n+1)\big)} t^{0.75\,k+1.5n}$$

And

$$L_2 z(t) \colon D_*^{1.5} z(t) + 2D_*^{0.75} z(t) + z(t)$$

$$\Longrightarrow p_2(s) = s^{1.5} + 2s^{0.75} + 1$$

$$\Longrightarrow \psi_2(s) = \frac{1}{p_2(s)} = \frac{1}{s^{1.5} + 2s^{0.75} + 1}$$

$$\Longrightarrow u_2(t) = \mathcal{L}^{-1}[\psi_2(s)] = t^{0.5} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^n (-2)^k \binom{n+k}{k}}{\Gamma\big(0.75\,k + 1.5(n+1)\big)} t^{0.75\,k+1.5n}$$

where the initial approximation must be satisfied by the following equations:

$$L_1 x(t) = 0, \qquad x(0) = 1, \acute{x}(0) = 1 \Longrightarrow x_0(t) = v_1^0$$

$$= \sum_{m=0}^{\infty} \frac{2^m}{m!} \Bigg[t^{1.5\,m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.75})^k}{\Gamma(0.75k + 0.75m + 0.75m + 1)} + t^{1.5\,m+0.75} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.75})^k}{\Gamma(0.75k + 0.75m + 0.75m + 1.75)} + t^{1.5\,m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.75})^k}{\Gamma(0.75k + 0.75m + 0.75m + 2)} \Bigg]$$

$$L_1 z(t) = 0, \qquad z(0) = 1, \acute{z}(0) = -1 \Longrightarrow z_0(t) = v_2^0$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Bigg[t^{1.5\,m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.75})^k}{\Gamma(0.75k + 0.75m + 0.75m + 1)} + t^{1.5\,m+0.75} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.75})^k}{\Gamma(0.75k + 0.75m + 0.75m + 1.75)} - t^{1.5\,m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.75})^k}{\Gamma(0.75k + 0.75m + 0.75m + 2)} \Bigg]$$

$$y_0(t) = \Bigg(\sum_{m=0}^{\infty} \frac{2^m}{m!} \Bigg[t^{1.5\,m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.75})^k}{\Gamma(0.75k + 0.75m + 0.75m + 1)} + t^{1.5\,m+0.75} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.75})^k}{\Gamma(0.75k + 0.75m + 0.75m + 1.75)} + t^{1.5\,m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.75})^k}{\Gamma(0.75k + 0.75m + 0.75m + 2)} \Bigg] \Bigg)^2$$

Accordingly, by Eq. (5.3) the higher order approximation of the exact solution can be obtained as follows:

$$x_1(t) = \sum_{i=0}^{1} v_1^k(t) = \sum_{m=0}^{\infty} \frac{2^m}{m!} \Bigg[t^{1.5\,m} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.75})^k}{\Gamma(0.75k + 0.75m + 0.75m + 1)} + t^{1.5\,m+0.75} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.75})^k}{\Gamma(0.75k + 0.75m + 0.75m + 1.75)} + t^{1.5\,m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(-t^{0.75})^k}{\Gamma(0.75k + 0.75m + 0.75m + 2)} \Bigg] + \int_0^t \left(\frac{(\tau - t)^{0.75}}{\Gamma(0.75+1)} \right) \left((t-\tau)^{0.5} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(2)^n (-1)^k \binom{n+k}{k}}{\Gamma\big(0.75\,k + 1.5(n+1)\big)} (t-\tau)^{0.75\,k+1.5n} \right)$$

$$\left(\begin{array}{c} 1-\left(\sum_{m=0}^{\infty}\frac{2^m}{m!}\left[\begin{array}{c} t^{1.5\,m}\sum_{k=0}^{\infty}\frac{(k+m)!}{k!}\frac{(-t^{0.75})^k}{\Gamma(0.75k+0.75m+0.75m+1)} \\ +t^{1.5\,m+0.75}\sum_{k=0}^{\infty}\frac{(k+m)!}{k!}\frac{(-t^{0.75})^k}{\Gamma(0.75k+0.75m+0.75m+1.75)} \\ +t^{1.5\,m+1}\sum_{k=0}^{\infty}\frac{(k+m)!}{k!}\frac{(-t^{0.75})^k}{\Gamma(0.75k+0.75m+0.75m+2)} \end{array}\right]\right) \\ \left(\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left[\begin{array}{c} t^{1.5\,m}\sum_{k=0}^{\infty}\frac{(k+m)!}{k!}\frac{(2t^{0.75})^k}{\Gamma(0.75k+0.75m+0.75m+1)} \\ +t^{1.5\,m+0.75}\sum_{k=0}^{\infty}\frac{(k+m)!}{k!}\frac{(2t^{0.75})^k}{\Gamma(0.75k+0.75m+0.75m+1.75)} \\ -t^{1.5\,m+1}\sum_{k=0}^{\infty}\frac{(k+m)!}{k!}\frac{(2t^{0.75})^k}{\Gamma(0.75k+0.75m+0.75m+2)} \end{array}\right]\right) \end{array}\right) d\tau$$

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Table 1. Numerical results of the solution in Example 5.1

T	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	Exact solution
0	1	1	1	1
0.1	1.085	1.09	1.105	1.105
0.2	1.215	1.216	1.221	1.221
0.3	1.262	1.311	1.35	1.35
0.4	1.292	1.451	1.492	1.492
0.5	1.359	1.451	1.649	1.649
0.6	1.428	1.554	1.822	1.822
0.7	1.501	1.666	2.014	2.014
0.8	1.581	1.789	2.226	2.226
0.9	1.668	1.924	2.46	2.46
1	1.765	2.071	2.718	2.718

Table 1 shows the approximate solutions for Eq. (5.5) obtained for different values of α using our method. The results are in good agreement with the results of the exact solutions.

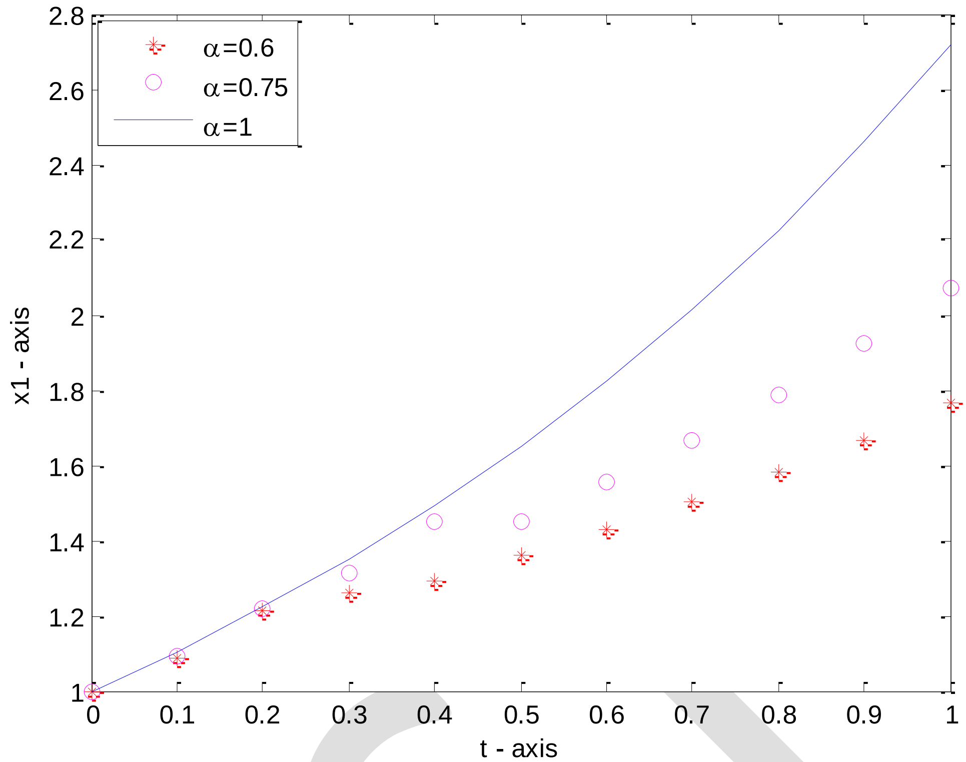

Fig.1 results for Example 1

Example 2: consider the following system of linear space-time fractional order differential – algebraic equations

$$\left.\begin{array}{r} D_*^{2\alpha}x(t) + 2D_*^{\alpha}x(t) + x(t) - y(t) + \cos t = 0, \\ y(t) + x(t) = te^{-t} + \cos t, \quad t \in [o,1], \quad 0 < \alpha \leq 1 \end{array}\right\} \quad \ldots(5.4)$$

Subject to initial conditions x(0) =0, $\acute{x}(0) = 1$, $y(0) = 1$. For the special case $\alpha = 1$, we have analytical solution $x(t) = te^{-t}$ and $y(t) = \cos t$.

Solution:

From the Eq. (5.4), optimal selection auxiliary linear operator the equation is represented as follows:
$Lx(t): D_*^{2\alpha}x(t) + 2D_*^{\alpha}x(t) + x(t)$

Therefore $\emptyset(v_1^k, v_2^k)$ is defined as:

$$\emptyset(v_1^k, v_2^k) = \int_0^t u(t-\tau)[v_2^k(\tau) - \cos\tau]d\tau \Big\} \quad \ldots(5.5)$$

Then, using Eq. (5.5), the Laplace Iteration Method with Fractional Lagrange Multiplier formulae in t-direction for the calculation of the approximate solution of Eq.(5.3) can be readily obtained as:

$$\left.\begin{array}{r} x_{n+1}(t) = x_0(t) + \int_0^t [\lambda(\tau)u(t-\tau)(y_n(\tau) - \cos\tau)\mathrm{d}\tau, \\ y_n(t) = te^{-t} + \cos t - x_n(t) \end{array}\right\} \quad \ldots(5.6)$$

<u>Case 1: $\alpha = 1$</u>

$$Lx(t): D_*^2x(t) + 2D_*^1x(t) + x(t)$$
$$\Rightarrow p(s) = s^2 + 2s + 1$$
$$\Rightarrow \psi(s) = \frac{1}{p(s)} = \frac{1}{s^2 + 2s + 1}$$
$$\Rightarrow u(t) = \mathcal{L}^{-1}[\psi(s)] = te^{-t}$$

where the initial approximation must be satisfied by the following equations:

$$L_1 x(t) = 0, \qquad x(0) = 0, \qquad \acute{x}(0) = 1 \Longrightarrow x_0(t) = v_1^0 = te^{-t}$$
$$y_0(t) = \cos t$$

Accordingly, by Eq. (5.6) the higher order approximation of the exact solution can be obtained as follows:

$$x_1(t) = \sum_{i=0}^{1} v_1^k(t) = te^{-t} + \int_0^t (\tau - t)(t - \tau)e^{(\tau - t)}(y_0(\tau) - \cos\tau)d\tau$$
$$y_1(t) = \cos t$$

The remaining approximations $x_n = 0, z_n = 0, n > 1$ can be completely determined such that each term will be determined using the prevous term: thus, the exact solution is as follows:

$$x(t) = \lim_{p\to\infty} \sum_{i=0}^{p} v_1^k(t) = te^{-t}$$
$$y(t) = \cos t$$

Case 2: $\alpha = 0.6$

$$Lx(t): D_*^{1.2}x(t) + 2D_*^{0.6}x(t) + x(t)$$
$$\Longrightarrow p(s) = s^{1.2} + 2s^{0.6} + 1$$
$$\Longrightarrow \psi(s) = \frac{1}{p(s)} = \frac{1}{s^{1.2} + 2s^{0..6} + 1}$$
$$\Longrightarrow u(t) = \mathcal{L}^{-1}[\psi(s)] = t^{0.2} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^n (-2)^k \binom{n+k}{k}}{\Gamma(0.6\,k + 1.2(n+1))} t^{0.6\,k+1.2n}$$

where the initial approximation must be satisfied by the following equations:

$$Lx(t) = 0, \qquad x(0) = 0, \qquad \acute{x}(0) = 1$$
$$x_0(t) = v_1^0 = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left[t^{1.2m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.6})^k}{\Gamma(0.6k + 1.2m + 2)}\right]$$

$$y_0(t) = \mathrm{t}e^{-t} + \cos(t) - \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left[t^{1.2m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.6})^k}{\Gamma(0.6k+1.2m+2)}\right]$$

Accordingly, by Eq. (5.6) the higher order approximation of the exact solution can be obtained as follows:

$$x_1(t) = \sum_{i=0}^{1} v_1^k(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left[t^{1.2m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.6})^k}{\Gamma(0.6k + 1.2m + 2)}\right]$$

$$\left((t-\tau)^{0.2} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-2)^n (-1)^k \binom{n+k}{k}}{\Gamma(0.6\,k + 1.2(n+1))} (t-\tau)^{0.6\,k+1.2n}\right)$$
$$(y_0(\tau) - \cos(\tau)]d\tau$$

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Case 1: $\alpha = 0.75$

$$Lx(t): D_*^{1.5}x(t) + 2D_*^{0.75}x(t) + x(t)$$
$$\Longrightarrow p(s) = s^{1.5} + 2s^{0.75} + 1$$
$$\Longrightarrow \psi(s) = \frac{1}{p(s)} = \frac{1}{s^{1.5} + 2s^{0.75} + 1}$$

$$\Rightarrow u(t) = \mathcal{L}^{-1}[\psi(s)] = t^{0.5} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^n\,(-2)^k\binom{n+k}{k}}{\Gamma\big(0.75\,k + 1.2(n+1)\big)} t^{0.75\,k+1.5n}$$

where the initial approximation must be satisfied by the following equations:

$$Lx(t) = 0, \qquad x(0) = 1, \acute{x}(0) = -1$$

$$x_0(t) = v_1^0 = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left[t^{1.5m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.75})^k}{\Gamma(0.75k + 1.5m + 2)}\right]$$

$$y_0(t) = \mathrm{t}e^{-t} + \cos(t) - \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left[t^{1.5m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(2t^{0.75})^k}{\Gamma(0.75k+1.5m+2)}\right]$$

Accordingly, by Eq. (5.6) the higher order approximation of the exact solution can be obtained as follows:

$$x_1(t) = \sum_{i=0}^{1} v_1^k(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\left[t^{1.5m+1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{\left(2t^{0.75}\right)^k}{\Gamma(0.75k + 1.5\,m + 2)}\right]$$

$$\left((t-\tau)^{0.5} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-2)^n\,(-1)^k\binom{n+k}{k}}{\Gamma\big(0.75\,k + 1.5(n+1)\big)} (t-\tau)^{0.75\,k+1.5n}\right)$$

$$(y_0(\tau) - \cos(\tau)]d\tau$$

.

Table 2. Numerical results of the solution in Example 5.2

T	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$	Exact solution
0	0	0	0	0
0.1	0.1	0.099	0.09	0.09
0.2	0.2	0.195	0.164	0.164
0.3	0.3	0.285	0.22	0.222
0.4	0.4	0.37	0.268	0.268
0.5	0.5	0.447	0.303	0.303
0.6	0.6	0.516	0.329	0.329
0.7	0.7	0.577	0.348	0.348
0.8	0.8	0.628	0.359	0.359
0.9	0.9	0.669	0.366	0.366
1	1	0.669	0.368	0.368

Table 2 shows the approximate solutions for Eq. (5.5) obtained for different values of α using our method. The results are in good agreement with the results of the exact solutions.

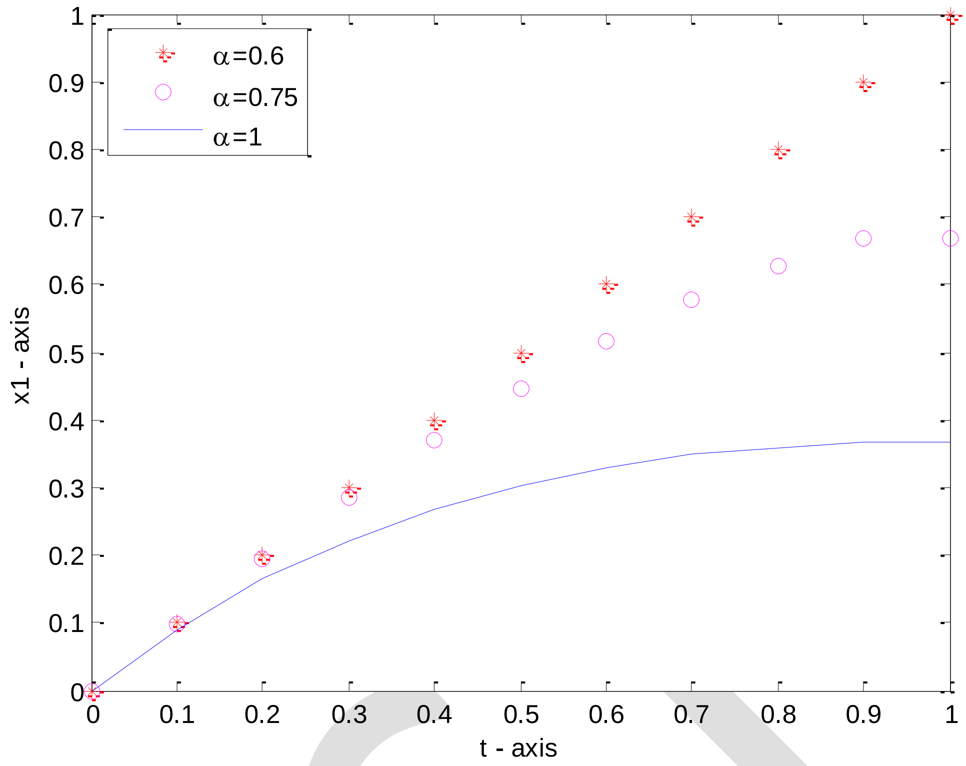

Fig.2 results for Example 2

Conclusions

In this paper, we have introduced a combination of the Laplace Iteration method and Variational Iteration method for multi-term fractional equations. This combination builds a strong method called Laplace Lagrange Iteration Method (LLIM). We used this method for solving the system of time-space fractional order of differential-algebraic equations.

The LLIM has been shown to solve effectively, easily and accurately large class of non-linear problems with the approximations which convergent are rapidly to exact solutions. Finally, we conclude that the LLIM may be considered as a nice refinement in existing numerical techniques.

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