t-REGULAR MODULES

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ABSTRACT

Let \( R \) be a commutative ring with identity and \( M \) be an \( R \)-module. Let \( Z_2(M) \) be the second singular submodule of \( M \). In this paper we introduce the concept of \( t \)-regular modules as a generalization of regular modules. The module \( M \) is called \( t \)-regular module if \( t_M(I) \) is a pure submodule of \( M \), for each ideal \( I \) of \( R \), where \( t_M(I) = \{ m \in M | \text{Im} \subseteq Z_2(M) \} \). Some properties of this class of modules are investigated and some relationships between these modules and other related modules are introduced.

Key words: Regular modules, \( t \)-Regular module, Pure submodules.

1. INTRODUCTION

Throughout this paper \( R \) denotes a commutative ring with identity, modules are unital \( R \)-modules. Let \( M \) be an \( R \)-module, the singular submodule of \( M \) is \( Z(M) = \{ m \in M | \text{Im} = 0 \text{ for some essential ideal I of } R \} \). If \( M = Z(M) \), then \( M \) is called singular and \( M \) is nonsingular provided \( Z(M) = 0 \). The second singular submodule, in other words, the Goldie torsion submodule \( Z_2(M) \) is defined by \( Z(M/Z(M)) = Z_2(M)/Z(M) \). The module \( M \) is called \( Z_2 \)-torsion (or Goldie torsion) if \( M = Z_2(M) \). It is clear that every singular module is \( Z_2 \)-torsion. Let \( R \) be an integral domain and \( M \) be an \( R \)-module. The torsion submodule of \( M \) is \( T(M) = \{ m \in M | rm = 0 \text{ for some } r \in R \setminus \{0\} \} \). If \( M = T(M) \), then \( M \) is called torsion and \( M \) is torsion-free, provided \( T(M) = 0 \). It is evident that the singular and nonsingular concepts for the modules over an integral domain are equivalent to the torsion and torsion-free respectively, in other words, \( Z(M) = T(M) \). Following P.M. Cohn [4], a submodule \( N \) of an \( R \)-module \( M \) is called pure if the sequence \( 0 \rightarrow E \otimes N \rightarrow E \otimes M \) is exact for all \( R \)-modules \( E \). Anderson and Fuller in [2] called the submodule \( N \) of an \( R \)-module \( M \) is pure if \( IM \cap N = IN \) for every ideal \( I \) of \( R \). It can easily see that the purity in the first definition implies to the second but not conversely. An \( R \)-module \( M \) is called regular if every submodule of \( M \) is (Cohn) pure [9].
In this paper, our aim is to introduce and study t-regular modules. An $R$-module $M$ is called t-regular module if $t_M(I)$ is a pure submodule of $M$ (in sense of Anderson and Fuller), for each ideal $I$ of $R$, where $t_M(I) = \{ m \in M \mid Im \subseteq Z_2(M) \}$. It is evident that every regular module is t-regular module, but not conversely (Remarks and Examples 2.2).

The work is structured in two sections. In section two we supply some elementary properties of t-regular modules. A characterization of t-regular modules is given (Theorem 2.4), we see that the t-regular modules have the pure intersection property for any finite collection of submodules $\{t_M(I_a)\}_{a=1}^n$, where $I_a$ is an ideal of $R$ (Proposition 2.6). We prove that a direct sum of two t-regular modules is also t-regular (Proposition 2.8). In section three, many relationships between t-regular modules and other related concepts are presented. We give a certain condition under which t-regular module is purely extending module (Proposition 3.7). Beside other results we see that if $M$ is t-regular $R$-module, then $t_M(I)$ is semiprime submodule of $M$ for each ideal $I$ of $R$ (Theorem 3.12). Also we show that every PF-ring (and hence PP-ring) is t-regular ring (Examples 3.13).

2. t-Regular Modules – Basic Results

In this section we introduce the concept of t-regular modules. The basic properties are investigated. It is shown that every direct sum of t-regular modules is again t-regular. We begin by giving our definition.

**Definition 2.1.** An $R$-module $M$ is called t-regular module if $t_M(I)$ is a pure submodule of $M$ (in sense of Anderson and Fuller), for each ideal $I$ of $R$, where $t_M(I) = \{ m \in M \mid Im \subseteq Z_2(M) \}$.

**Remarks and Examples 2.2.**

1. Every $Z_2$-torsion (singular) module is t-regular module since $Z_2(M) = M$ ($Z(M) = M$), then $t_M(I) = M$ is pure in $M$. The converse is not true in general. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is t-regular since $Z_2(\mathbb{Z}) = Z(\mathbb{Z}) = 0$ is pure in $\mathbb{Z}$ but $\mathbb{Z}$ is neither $Z_2$-torsion nor singular module.

2. Clearly that every regular module is t-regular module, but the converse is not true. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ is t-regular module but not regular. Also, one can easily see that the $\mathbb{Z}$-module $\mathbb{Z}_4$ is $Z_2$-torsion then by Remark and Example (1), $\mathbb{Z}_4$ is t-regular but it is not regular as $\mathbb{Z}$-module.

3. If $M$ is t-regular $R$-module, then the submodule $Z_2(M)$ is a pure submodule.

**Proof.** Since $t_M(R) = \{ m \in M \mid Rm \subseteq Z_2(M) \} = \{ m \in M \mid m \in Z_2(M) \} = Z_2(M)$ and $t_M(R)$ is pure submodule.
(4) For any $R$-module $M$, $t_M(\langle 0 \rangle) = \{ m \in M \mid \langle 0 \rangle m \leq Z_2(M) \} = M$ is a pure submodule in $M$.

(5) If $I_1$ and $I_2$ are ideals of a ring $R$ with $I_1 \leq I_2$, then $t_M(I_2) \leq t_M(I_1)$.

Proof. Since $m \in t_M(I_2)$, then $m \in M$, $I_2m \leq Z_2(M)$. But $I_1 \leq I_2$, then $I_1m \leq I_2m$. It follows that $I_1m \leq Z_2(M)$, that is $m \in t_M(I_2)$.

(6) It is clear for each ideal $I$ of a ring $R$, $Z_2(M) \subseteq t_M(I)$.

(7) Let $M$ be an $R$-module. If for each $N \leq M$, there exists an ideal $I$ of $R$ such that $N = t_M(I)$. Then $M$ is regular if and only if $M$ is $t$-regular.

(8) If $M$ is an $R$-module, and $N \leq M$. Then $N$ need not be written of the form $N = t_M(I)$ for any ideal $I$ of $R$. For example, the $\mathbb{Z}$-module $\mathbb{Z}_4$ where $N = \{ \overline{0}, \overline{2} \}$ is the submodule generated by $\overline{2}$. Since $Z_2(\mathbb{Z}_4) = \mathbb{Z}_4$, one can easily see that $t_{\mathbb{Z}_4}(I) = \mathbb{Z}_4$, for each ideal $I$ of $\mathbb{Z}$. Moreover this example shows that the condition in Remark (7) that each submodule $N$ of $M$ is written of the form $t_M(I)$ for some ideal $I$ of $R$ is necessary to make the t-regular module is regular module.

Before we give our next result we need the following lemma

**Lemma 2.3.** If $M$ is an $R$-module, then for each ideal $I$ of $R$, and for each $a \in I$, $t_M(I) = \bigcap_{a \in I} t_M(\langle a \rangle)$.

**Proof.** Let $a \in I$ and $m \in t_M(I)$, $\langle a \rangle m \subseteq I m \subseteq Z_2(M)$. Then $m \in t_M(\langle a \rangle)$ for each $a \in I$, implies that $m \in \bigcap_{a \in I} t_M(\langle a \rangle)$. For the reverse inclusion, let $m \in \bigcap_{a \in I} t_M(\langle a \rangle)$, then $m \in t_M(\langle a \rangle)$, $\langle a \rangle m \subseteq Z_2(M)$. So $am \in Z_2(M)$ for each $a \in I$, it follows that $Im \subseteq Z_2(M)$. Therefore $m \in t_M(I)$.

Note that if $I$ is generated by the finite set $\{ a_1, a_2, \ldots, a_n \}$, $I = Ra_1 + Ra_2 + \ldots + Ra_n$. Then by lemma 2.3, we have $t_M(I) = t_M(\sum_{i=1}^n Ra_i) = \bigcap_{i=1}^n t_M(Ra_i)$.

**Theorem 2.4.** Let $M$ be an $R$-module. Then $M$ is $t$-regular module if and only if $t_M(I)$ is a pure submodule of $M$ for each principal ideal $I$ of $R$.

**Proof.** ($\Rightarrow$) It is clear since $M$ is $t$-regular module.

($\Leftarrow$) Let $t_M(I)$ be a pure submodule for each principal ideal $I$ of $R$. To prove that $M$ is $t$-regular module. Let $K$ be an ideal of $R$, we have to show that $JM \cap t_M(K) = J t_M(K)$, for each ideal $J$ of $R$. Let $m \in JM \cap t_M(K)$, then $m = \sum_{i=1}^n b_i m_i$, where $b_i \in J$, $m_i \in M$ and $km \subseteq Z_2(M)$. So $am \in Z_2(M)$, for each $a \in K$. It follows that $m \in t_M(\langle a \rangle)$, and hence $m \in JM \cap t_M(\langle a \rangle)$. But $t_M(\langle a \rangle)$ is a pure submodule in $M$, then $JM \cap t_M(\langle a \rangle) = J t_M(\langle a \rangle)$. That is $m \in J t_M(\langle a \rangle)$. Assume $m = \sum_{i=1}^n b_i' m_i'$, where $b_i' \in J$ and $m_i' \in t_M(\langle a \rangle)$, for each $a \in K$. Thus $m_i' \in \bigcap_{a \in K} t_M(\langle a \rangle)$. But $\bigcap_{a \in K} t_M(\langle a \rangle) = t_M(K)$ by lemma 2.3, then $m_i' \in t_M(K)$.

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therefore \( t_M(K) \) is a pure submodule of \( M \). It follows that \( M \) is t-regular module.

**Lemma 2.5.** Let \( M \) be an R-module. Then for any collection of ideals \( \{ I_a \}_{a \in \Lambda} \) of \( R \). Then \( t_M(\sum_{a \in \Lambda} I_a ) = \bigcap_{a \in \Lambda} t_M( I_a ) \).

**Proof.** Since for each \( a \in \Lambda \), \( I_a \subseteq \sum_{a \in \Lambda} I_a \), then by Remark and Example 2.2 (5), \( t_M(\sum_{a \in \Lambda} I_a ) \subseteq t_M( I_a ) \). Thus \( t_M(\sum_{a \in \Lambda} I_a ) \subseteq \bigcap_{a \in \Lambda} t_M( I_a ) \). Let \( m \in \bigcap_{a \in \Lambda} t_M( I_a ) \), then \( m \in t_M( I_a ) \), for each \( a \in \Lambda \). Hence \( I_a m \subseteq Z_2(M) \), implies that \( (\sum_{a \in \Lambda} I_a ) m \subseteq Z_2(M) \). That is, \( t_M(\sum_{a \in \Lambda} I_a ) = \bigcap_{a \in \Lambda} t_M( I_a ) \).

Recall that an R-module \( M \) is said to have the **pure intersection property** if the intersection any two pure submodules of \( M \) is again pure [1].

**Proposition 2.6.** Every t-regular R-module has the pure intersection property for any finite collection of submodules \( \{ t_M( I_a ) \}_{a} \), where \( I_a \) is an ideal of \( R \).

**Proof.** Let \( \{ t_M( I_a ) \}_{a} \) be a finite collection of submodules of \( M \), then by Lemma 2.5, \( t_M(\sum_{a=1}^{n} I_a ) = \bigcap_{a=1}^{n} t_M( I_a ) \). But \( M \) is t-regular, then \( t_M(\sum_{a=1}^{n} I_a ) \) is a pure submodule in \( M \). It follows that \( \bigcap_{a=1}^{n} t_M( I_a ) \) is a pure submodule in \( M \), that is \( M \) has the pure intersection property for \( \{ t_M( I_a ) \}_{a=1}^{n} \).

**Theorem 2.7.** Let \( M = M_1 \oplus M_2 \) where \( M_1 \) and \( M_2 \) be two R-modules. Then for each ideal \( I \) of \( R \), \( t_M(I) = t_{M_1}(I) \oplus t_{M_2}(I) \).

**Proof.** Let \( m \in t_M(I) \), then \( m \in M = M_1 \oplus M_2 \). Let \( m = (m_1, m_2) \) where \( m_1 \in M_1 \), \( m_2 \in M_2 \). So \( I(m_1, m_2) \subseteq Z_2(M_1 \oplus M_2) = Z_2(M_1) \oplus Z_2(M_2) \), implies that \( Im_1 \subseteq Z_2(M_1) \) and \( Im_2 \subseteq Z_2(M_2) \). Thus \( m_1 \in t_{M_1}(I) \) and \( m_2 \in t_{M_2}(I) \), it follows that \( (m_1, m_2) \in t_{M_1}(I) \oplus t_{M_2}(I) \). For the reverse inclusion, let \( m \in t_{M_1}(I) \oplus t_{M_2}(I) \). So \( m = (m_1, m_2) \in t_{M_1}(I) \oplus t_{M_2}(I) \), \( m_1 \in t_{M_1}(I) \) and \( m_2 \in t_{M_2}(I) \). Thus \( Im_1 \subseteq Z_2(M_1) \) and \( Im_2 \subseteq Z_2(M_2) \), that is \( I(m_1, m_2) = Im_1 \oplus Im_2 = Z_2(M_1) \oplus Z_2(M_2) = Z_2(M_1 \oplus M_2) \). Therefore \( m \in t_M(I) \).

**Proposition 2.8.** Let \( M = M_1 \oplus M_2 \) where \( M_1 \) and \( M_2 \) be two R-modules. Then \( M_1 \) and \( M_2 \) are t-regular modules if and only if \( M \) is t-regular.

**Proof.** (\( \Rightarrow \)) Assume \( M = M_1 \oplus M_2 \) where \( M_1 \) and \( M_2 \) are t-regular R-modules. Let \( I \) and \( J \) be ideals of \( R \). To show that \( JM \cap t_M(I) = J t_M(I) \). Since \( J M \cap \)
\[ t_M(I) = J(M_1 \oplus M_2) \cap (t_{M_1}(I) \oplus t_{M_2}(I)) \] by Theorem 2.7. So after simple steps, one can easily see that, \( J M \cap t_M(I) = J t_M(I) \). That is \( t_M(I) \) is pure submodule in \( M \).

\[(\Leftarrow)\text{ Assume } M = M_1 \oplus M_2 \text{ is } t\text{-regular } R\text{-module, then } J M \cap t_M(I) = J t_M(I) \text{ for each ideals } I \text{ and } J \text{ of } R. \text{ Since } J M \cap t_M(I) = J(M_1 \oplus M_2) \cap t_{M_1}(I) \oplus t_{M_2}(I) = J(t_{M_1}(I) \oplus t_{M_2}(I)), \text{ hence } J M_1 \cap t_{M_1}(I) = J t_{M_1}(I) \text{ and } J M_2 \cap t_{M_2}(I) = J t_{M_2}(I). \text{ Hence } M_1 \text{ and } M_2 \text{ are } t\text{-regular modules.}

**Corollary 2.9.** Every direct summand of \( t\text{-regular module is also } t\text{-regular.}

**Proof.** It follows by Proposition 2.8.

We end the section by the following example

**Example 2.10.** Consider the module \( \mathbb{Z}_4 \) as \( \mathbb{Z}_4\)-module, it is not hard to see that \( \mathbb{Z}_2 (\mathbb{Z}_4) = \{0, \overline{2}\} \) is the submodule generated by \( \overline{2} \). Then \( t_{\mathbb{Z}_4} (\mathbb{Z}_4) = \{0, \overline{2}\} \) is not pure submodule in \( \mathbb{Z}_4 \) and hence \( \mathbb{Z}_4 \) is not \( t\)-regular module. Moreover by Proposition 2.8, the module \( \mathbb{Z}_4 \oplus \mathbb{Z}_4 \) is also not \( t\)-regular as \( \mathbb{Z}_4\)-module.

### 3. \( t\)-Regular Modules and Other Related Concepts

In this section we investigate the relationships between \( t\)-regular modules and some other modules such as nonsingular, CLS, purely extending, projective, and injective modules.

**Proposition 3.1.** Every nonsingular module over an integral domain is \( t\text{-regular module.}

**Proof.** Let \( M \) be a nonsingular module over an integral domain \( R \) and \( I \) be an ideal of \( R \). Then \( Z_2 (M) = 0 \), implies that \( t_M(I) = \{ m \in M \mid Im = 0 \} \subseteq T(M) = \{ m \in M \mid rm = 0 \} \text{ for some } r \in R \setminus \{0\} \}. \text{ But } T(M) = Z(M) = 0, \text{ then } t_M(I) = 0. \text{ That is } t_M(I) \text{ is a pure submodule of } M, \text{ and hence } M \text{ is } t\text{-regular.}

**Corollary 3.2.** Every an integral domain is a \( t\text{-regular.}

**Proof.** It follows directly by Proposition 3.1.

Recall that a submodule \( A \) of an \( R\)-module \( M \) is called \( y\)-closed if \( M /A \) is nonsingular module [10]. If every \( y\)-closed submodule of \( M \) is a direct summand, then \( M \) is said to be a **CLS-module** [14].
**Proposition 3.3.** Let $M$ be a CLS-module over an integral domain $R$, then $M$ is $t$-regular module.

**Proof.** Let $M$ be a CLS-module over an integral domain $R$, then every $y$-closed submodule is a direct summand of $M$. Since $Z_2(M)$ is $y$-closed submodule, then $M = Z_2(M) \oplus M'$ for some submodule $M'$ of $M$. It follows that $M/Z_2(M) \cong M'$ and since $M/Z_2(M)$ is nonsingular, so $M'$ is nonsingular $R$-module. Then $t_{M'}(I) = \{ m \in M' \mid \lim = 0 \} \subseteq T(M') = 0$, so $t_{M'}(I) = 0$. That is $t_{M'}(I)$ is a pure submodule in $M'$ and $t_{Z_2(M)}(I) = Z_2(M)$ is a pure submodule in $Z_2(M)$. Then by [1, Lemma 4.2], the direct sum of pure submodules is again pure, thus $t_{Z_2(M)}(I) \oplus t_{M'}(I)$ is a pure submodule in $Z_2(M) \oplus M' = M$. But $t_{Z_2(M)}(I) \oplus t_{M'}(I) = t_{M}(I)$ by Theorem 2.7, therefore $t_{M}(I)$ is a pure submodule in $M$, and hence $M$ is $t$-regular module.

**Proposition 3.4.** Let $M$ be an $R$-module. If $Z_2(M)$ is direct summand and maximal submodule, then $M$ is $t$-regular.

**Proof.** Since $Z_2(M) \leq t_{M}(I)$ for each ideal $I$ of $R$ and $Z_2(M)$ is maximal submodule, it follows that $Z_2(M) = t_{M}(I)$. But $Z_2(M)$ is a direct summand of $M$, this implies $t_{M}(I)$ is a direct summand of $M$. Hence $M$ is $t$-regular.

Recall that a submodule $A$ of an $R$-module is called an *essential* of $M$ (or $M$ is an *essential extension* of $A$) if $A \cap B \neq 0$, for every submodule $B$ of $M$. If $A$ has no proper essential extension in $M$, then $A$ is said to be *closed* [7] and $M$ is called an *extending module* (or *CS-module*) if every closed submodule of $M$ is a direct summand [7].

**Proposition 3.5.** Let $M$ be an $R$-module. If $M$ is CLS and every nonsingular submodule of $M$ is a closed submodule. Then $M$ is a $t$-regular module.

**Proof.** Let $I$ be an ideal of $R$. Since $M$ is CLS, then by [14, Proposition 8], $M = Z_2(M) \oplus M'$ for some submodule $M'$ of $M$ and $M'$ is a CS-module. Then $t_{M}(I) = t_{Z_2(M)}(I) \oplus t_{M'}(I)$ and $t_{M'}(I)$ is nonsingular submodule in $M'$ since $M/Z_2(M) \cong M'$, and $Z_2(M)$ is $y$-closed, $M/Z_2(M)$ is nonsingular. This implies that $t_{M'}(I)$ is nonsingular submodule in $M$. Then by hypothesis, $t_{M'}(I)$ is closed submodule in $M$. It follows that $t_{M'}(I)$ is closed submodule in $M'$. But $M'$ is CS-module, then $t_{M'}(I)$ is direct summand submodule in $M'$. Therefore $t_{M'}(I)$ is a pure submodule in $M'$ and $t_{Z_2(M)}(I) = Z_2(M)$ is a pure submodule in $Z_2(M)$. Then $t_{Z_2(M)}(I) \oplus t_{M'}(I)$ is a pure submodule in $Z_2(M) \oplus M' = M$. But $t_{Z_2(M)}(I) \oplus t_{M'}(I) = t_{M}(I)$ by Theorem 2.7, therefore $t_{M}(I)$ is a pure submodule in $M$, and hence $M$ is a $t$-regular module.
Recall that an R-module $M$ is called a purely extending module if every closed submodule in $M$ is a pure submodule in $M$ [4].

**Proposition 3.6.** Let $M$ be an R-module such that $Z_2(M)$ is direct summand and every nonsingular submodule of $M$ is a closed submodule. If $M$ is purely extending module, then $M$ is $t$-regular.

**Proof.** By the same argument of the proof of Proposition 3.5.

**Proposition 3.7.** Let $M$ be a $t$-regular R-module. If for each closed submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = t_M(I)$. Then $M$ is a purely extending module.

**Proof.** Assume that $N$ is a closed submodule of $M$, then by hypothesis there exists an ideal $I$ of $R$ such that $N = t_M(I)$. Since $M$ is $t$-regular, then $t_M(I)$ is a pure submodule in $M$. Thus every closed submodule is pure and hence $M$ is purely extending.

**Proposition 3.8.** Let $M$ be an $R$-module such that $M = Z_2(M) \oplus M'$ for some submodule $M'$ of $M$. If $M'$ is a regular module, then $M$ is a $t$-regular module.

**Proof.** Since $M = Z_2(M) \oplus M'$, then $t_M(I) = t_{Z_2(M)}(I) \oplus t_{M'}(I)$. Assume that $M'$ is regular module, then $t_{M'}(I)$ is a pure submodule in $M'$ and $t_{Z_2(M)}(I) = Z_2(M)$ is a pure submodule in $Z_2(M)$. Hence $t_M(I)$ is a pure submodule in $M$ and thus $M$ is $t$-regular module.

Recall that an $R$-module $M$ is called $A$-projective, where $A$ is an $R$-module, if for each submodule $X$ of $A$, every homomorphism $h : M \to A / X$ can be lifted to a homomorphism $g : M \to A$. If $M$ is $A$-projective for every modules $A$, then $M$ is said to be a projective module [7].

**Proposition 3.9.** Let $R$ be a ring. The following statements are equivalent.

1. $\oplus_\Lambda R$ is $t$-regular $R$-module for each index set $\Lambda$.
2. Every projective $R$-module is $t$-regular module.

**Proof.** (1) $\Rightarrow$ (2) Let $M$ be a projective $R$-module, then there exists a free $R$-module $F$ and an $R$-epimorphism $f : F \to M$, and $F \cong \oplus_\Lambda R$ where $\Lambda$ is an index set. We have the following short exact sequence $0 \to \text{Ker} f \to \oplus_\Lambda R \xrightarrow{f} M \to 0$ where $i$ is the inclusion mapping. Since $M$ is projective, the sequence is split implies that $\oplus_\Lambda R \cong \text{Ker} f \oplus M$. But $\oplus_\Lambda R$ is $t$-regular $R$-module. Therefore by Proposition 2.8, $M$ is $t$-regular module.
(2) $\Rightarrow$ (1) It is clear.

Recall that an $R$-module $M$ is called $A$-injective where $A$ is an $R$-module, if for each submodule $X$ of $A$, every homomorphism $h : X \rightarrow M$, can be extended to a homomorphism $g : A \rightarrow M$. If $M$ is $A$-injective for every modules $A$, then $M$ is said to be an injective module [7].

**Proposition 3.10.** Let $R$ be a ring. Consider the following statements.

1. $R$ is semisimple ring.
2. Every $R$-module is injective.
3. Every $R$-module is $t$-regular.
4. Every $R$-module is projective.

**Proof.** $(1) \iff (2)$ and $(1) \iff (4)$ by [12, Corollary 8.2.2].

$(2) \Rightarrow (3)$ Let $M$ be an injective $R$-module. Since every $R$-module is injective, then $Z_2(M)$ is an injective submodule of $M$. Then by [1, Remark 1.3] $Z_2(M)$ is direct summand of $M$. Thus $M = Z_2(M) \oplus M'$ for some submodule $M'$ of $M$, which implies that $t_M(I) = t_{Z_2(M)}(I) \oplus t_{M'}(I)$. Again by hypothesis, $t_{M'}(I)$ is an injective submodule of $M$, it follows that $t_{M'}(I)$ is direct summand of $M$. Thus $t_{M'}(I)$ is a pure submodule in $M'$ and $t_{Z_2(M)}(I) = Z_2(M)$ is a pure submodule in $Z_2(M)$. Thus $M$ is $t$-regular module.

Recall that an $R$-module $M$ is called multiplication if for each submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$. Equivalently $M$ is a multiplication if for each submodule $N$ of $M$, $N = (N :_R M)M$ where $(N :_R M) = \{ r \in R \mid rM \subseteq N \}$ [8].

**Proposition 3.11.** Let $M$ be a finitely generated faithful multiplication $R$-module. If $Z_2(M) \subseteq Z_2(R)M$. Then $R$ is $t$-regular if and only if $M$ is $t$-regular module.

**Proof.** $( \Rightarrow )$ Let $R$ be a $t$-regular ring and $I$ be an ideal of $R$. To prove that $t_M(I)$ is a pure submodule of $M$. Since $M$ is a finitely generated faithful multiplication then by [8, Theorem 3.1], there exists a unique ideal $J$ of $R$ such that $t_M(I) = JM$ and one can easily see that $J = [ t_M(I) :_R M ]$. We claim that $[ t_M(I) :_R M ] = t_R(I)$. To show this, let $r \in [ t_M(I) :_R M ]$, implies that $rM \subseteq t_M(I)$. That is $rIM \subseteq Z_2(M) \subseteq Z_2(R)M$. Then $rI \subseteq Z_2(R)$, that is $r \in t_R(I)$. By the same argument one can prove the reverse inclusion, it follows that $t_M(I) = [ t_M(I) :_R M ]M = t_R(I)M$. But $R$ is $t$-regular, then $t_R(I)$ is a pure ideal of $R$. Since $M$ is finitely generated faithful multiplication, it is not hard to see that $t_R(I)M$ is a pure submodule of $M$. Therefore $t_M(I)$ is a pure submodule of $M$ and hence $M$ is $t$-regular module.
(⇐) It follows by similar proof.

Recall that a proper submodule $N$ of an $R$-module $M$ is called a semiprime submodule if for every $r \in R, x \in M, k \in \mathbb{Z}^+$ such that $r^kx \in N$ implies $rx \in N$ [6]. Equivalently, a proper submodule $N$ of an $R$-module $M$ is called semiprime if for each $r \in R, x \in M$ with $r^2x \in N$, implies that $rx \in N$ [3].

**Theorem 3.12.** If $M$ is $t$-regular $R$-module, then $t_M(I)$ is semiprime submodule of $M$ for each ideal $I$ of $R$.

**Proof.** Let $r \in R$ and $m \in M$ such that $r^2m \in t_M(I)$. Then $Ir^2m \in Z_2(M)$, that is $rIrM \in Z_2(M)$, implies that $rm \in t_M(rI)$. Since $rm \in rM$, then $rm \in rM \cap t_M(rI) = rM(rI)$. Thus $rm = rm'$ for some $m' \in t_M(rI)$, and hence $IrM = IrM' \subseteq Z_2(M)$. It follows that $rm \in t_M(I)$.

We end our work by the following examples.

**Examples 3.13.**

1. Every PF-ring $R$ (and hence PP-ring ) is $t$-regular ring where a ring $R$ is called PF if for each $a \in R$, $ann_R(a)$ is a pure ideal of $R$ [11]. If for each $a \in R$, $ann_R(a)$ is a direct summand of $R$ then $R$ is said to be PP-ring [15]. One can easily check that if $R$ is PF (or PP-ring ), then $R$ is a nonsingular, it follows that $ann_R(a) = t_R(a)$. Therefore $R$ is a $t$-regular ring.

2. Let $R$ be an integral domain, then $\oplus_\lambda R$ is $t$-regular as $R$-module since $Z_2(\oplus_\lambda R) = \oplus_\lambda Z_2(R)$ by [13, Proposition 2.2.3]. But $Z_2(R) = 0$, then $Z_2(\oplus_\lambda R) = 0$. Therefore $\oplus_\lambda R$ is nonsingular as $R$-module. Hence by Proposition 3.1, $\oplus_\lambda R$ is $t$-regular as $R$-module.

**References**


