On Generalized Jordan $(\sigma, \tau)$-Higher Homomorphisms of $\Gamma M$-Module

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Abstract:
Let $M$ be a $\Gamma$-ring and $X$ be a left $\Gamma M$-module, in this paper proved that every generalized Jordan $(\sigma, \tau)$-higher homomorphism from a $\Gamma$-ring $M$ into a prime left $\Gamma M$-module $X$ is either generalized $(\sigma, \tau)$-higher homomorphism or $(\sigma, \tau)$-higher anti homomorphism.

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Key Words: $\Gamma$-ring, left $\Gamma M$-module, generalized homomorphism, generalized Jordan homomorphism

1. Introduction
Let $M$ and $\Gamma$ be two additive abelian groups, suppose that there is a mapping from $M \times \Gamma \times M \longrightarrow M$ (the image of $(a, \alpha, b)$ being denoted by $a\alpha b$, $a, b \in M$ and $\alpha \in \Gamma$) satisfying for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$

$\alpha(\alpha + \beta)c = a\alpha c + ab\alpha c$

$a\alpha(b + c) = a\alpha b + a\alpha c$

(ii) $(a\alpha b)\beta c = a\alpha(\beta c)$.

Then $M$ is called $\Gamma$-ring. This definition is due to Barnes [1].

Let $M$ be a $\Gamma$-ring and $X$ be an additive abelian group. $X$ is a left $\Gamma M$-module if there exists a mapping $M \times \Gamma \times X \longrightarrow X$ (sending $(m, \alpha, x) \longrightarrow m \alpha x$), such that

(i) $(m_1 + m_2)\alpha x = m_1\alpha x + m_2\alpha x$

(ii) $m\alpha(x_1 + x_2) = m\alpha x_1 + m\alpha x_2$

(iii) $(m_1 \alpha m_2)\beta x = m_1\alpha(m_2\beta x)$

For all $m, m_1, m_2 \in M$ and $x, x_1, x_2 \in X$ and $\alpha, \beta \in \Gamma$, [5].

A $\Gamma$-ring $M$ is commutative if $a\alpha b = b\alpha a$ [7].

$X$ is prime if $a\Gamma X b = 0$ implies $a = 0$ or $b = 0$, for all $x \in X$ and $X$ is semiprime if $a\Gamma X \Gamma a = 0$ implies $a = 0$, for all $x \in X$.

$X$ is called a 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in X$ [5].

Let $X$ be a 2-torsion free semiprime $\Gamma M$-module $X$ and suppose that $a, b \in \Gamma M$-module $X$ if $a\Gamma X \Gamma b + b\Gamma X \Gamma a = 0$ for all $x \in X$, then $a\Gamma X b = b\Gamma X a = 0$.

Let $M$ be $\Gamma$-ring, a mapping $*: M \longrightarrow X$ is called an involution if for all $a, b \in M$ and $\alpha \in \Gamma$

(i) $a^{**} = a$.

(ii) $(a + b)^* = a^* + b^*$
(iii) \((a\sigma b)^* = b^s\alpha a^s\), [6].

Let \(\theta\) be an additive mapping of a ring \(R\) into a ring \(R'\), \(\theta\) is called a homomorphism if \(\theta(a\ b) = \theta(a) \theta(b)\).
And \(\theta\) is called a Jordan homomorphism if for all \(a, b \in R\)
\[\theta(a\ b + b\ a) = \theta(a) \theta(b) + \theta(b) \theta(a)\] for all \(a, b \in R\), [2].

An additive mapping \(F\) of a ring \(R\) into a ring \(R'\) is called a generalized homomorphism if there exists a homomorphism \(\theta\) from a ring \(R\) into a ring \(R'\) such that
\[F(ab) = F(a)\theta(b)\], for all \(a, b \in R\).
where \(\theta\) is called the relating homomorphism.

And \(F\) called a generalized Jordan homomorphism if there exists a Jordan homomorphism \(\theta\) from a ring \(R\) into a ring \(R'\) such that
\[F(ab + ba) = F(a)\theta(b) + F(b)\theta(a)\], for all \(a, b \in R\), where \(\theta\) is called the relating Jordan homomorphism.

Let \(\theta\) be an additive mapping of a \(\Gamma\)-ring \(M\) into a \(\Gamma\)-ring \(M'\), \(\theta\) is called homomorphism if
\[\theta(aabb) = \theta(a)\alpha\theta(b)\], for all \(a, b \in M\) and \(\alpha \in \Gamma\). [1].

Let \(\theta\) be an additive mapping of a \(\Gamma\)-ring \(M\) into a \(\Gamma\)-ring \(M'\), \(\theta\) is called Jordan homomorphism if
\[\theta(aabb + baab) = \theta(a)\alpha\theta(b) + \theta(b)\alpha\theta(a)\], for all \(a, b \in M\) and \(\alpha \in \Gamma\), [8].

An additive mapping \(F\) of a \(\Gamma\)-ring \(M\) into a \(\Gamma\)-ring \(M'\) is called generalized homomorphism if there exists a homomorphism \(\theta\) from a \(\Gamma\)-ring \(M\) into a \(\Gamma\)-ring \(M'\) such that
\[F(ab) = F(a)\alpha\theta(b)\], for all \(a, b \in M\) and \(\alpha \in \Gamma\).
where \(\theta\) is called the relating homomorphism, [8].

And \(F\) is called a generalized Jordan homomorphism if there exists a Jordan homomorphism \(\theta\) from a \(\Gamma\)-ring \(M\) into a \(\Gamma\)-ring \(M'\) such that
\[F(ab + ba) = F(a)\alpha\theta(b) + F(b)\alpha\theta(a)\], for all \(a, b \in M\) and \(\alpha \in \Gamma\), [8].
Where \(\theta\) is called the relating Jordan homomorphism.

Let \(\theta\) be an additive mapping of a ring \(R\) into a ring \(R'\) and \(\sigma, \tau\) be two endomorphisms of \(R\). \(\theta\) is called \((\sigma, \tau)\)-homomorphism if
\[\theta(ab) = \theta(\sigma(a)) \theta(\tau(b))\], for all \(a, b \in R\).

And \(\theta\) is called Jordan \((\sigma, \tau)\)-homomorphism if
\[\theta(ab + ba) = \theta(\sigma(a)) \theta(\tau(b)) + \theta(\sigma(b)) \theta(\tau(a))\], for all \(a, b \in R\), [3].

Let \(F\) be an additive mapping of a ring \(R\) into a ring \(R'\) and \(\sigma, \tau\) be two endomorphisms of \(R\). \(F\) is called a generalized \((\sigma, \tau)\)-homomorphism if there exists a \((\sigma, \tau)\)-homomorphism \(\theta\) from a ring \(R\) into a ring \(R'\) such that
\[F(ab) = F(\sigma(a)) \theta(\tau(b))\], for all \(a, b \in R\).
Where \(\theta\) is called the relating \((\sigma, \tau)\)-homomorphism.

And \(F\) is called generalized Jordan \((\sigma, \tau)\)-homomorphism if there exists a Jordan \((\sigma, \tau)\)-homomorphism \(\theta\) from a ring \(R\) into a ring \(R'\) such that
\[F(ab + ba) = F(\sigma(a)) \theta(\tau(b)) + F(\sigma(b)) \theta(\tau(a))\], for all \(a, b \in R\).
Where \(\theta\) is called the relating Jordan \((\sigma, \tau)\)-homomorphism, [3].
Let \( \theta \) be an additive mapping of a \( \Gamma \)-ring \( M \) into a \( \Gamma \)-ring \( M' \) and \( \sigma, \tau \) be two endomorphisms of \( M \). \( \theta \) is called \((\sigma, \tau)\)-homomorphism if
\[
\theta(ab) = \theta(a) \theta(b), \quad \text{for all } a, b \in M \text{ and } \alpha \in \Gamma.
\]

And \( \theta \) is called Jordan \((\sigma, \tau)\)-homomorphism if
\[
\theta(ab + ba) = \theta(a) \theta(b) + \theta(b) \theta(a), \quad \text{for all } a, b \in M \text{ and } \alpha \in \Gamma, \quad [3].
\]

Let \( F \) be an additive mapping of a \( \Gamma \)-ring \( M \) into a \( \Gamma \)-ring \( M' \) and \( \sigma, \tau \) be two endomorphisms of \( M \). \( F \) is called a generalized \((\sigma, \tau)\)-homomorphism if there exists a \((\sigma, \tau)\)-homomorphism \( \theta \) from a \( \Gamma \)-ring \( M \) into a \( \Gamma \)-ring \( M' \) such that
\[
F(ab) = F(a)F(b), \quad \text{for all } a, b \in M \text{ and } \alpha \in \Gamma.
\]

Where \( \theta \) is called the relating \((\sigma, \tau)\)-homomorphism.

And \( F \) is called generalized Jordan \((\sigma, \tau)\)-homomorphism if there exists a Jordan \((\sigma, \tau)\)-homomorphism \( \theta \) from a \( \Gamma \)-ring \( M \) into a \( \Gamma \)-ring \( M' \) such that
\[
F(ab + ba) = F(a)F(b) + F(b)F(a), \quad \text{for all } a, b \in M \text{ and } \alpha \in \Gamma.
\]

Where \( \theta \) is called the relating Jordan \((\sigma, \tau)\)-homomorphism, [3].

Now, in this paper we present the definitions of generalized \((\sigma, \tau)\)-higher homomorphism, generalized Jordan \((\sigma, \tau)\)-higher homomorphism, generalized Jordan triple \((\sigma, \tau)\)-higher homomorphism on a left \( \Gamma M \)-module and prove that every generalized Jordan \((\sigma, \tau)\)-higher homomorphism from a \( \Gamma \)-ring \( M \) into 2-torsion free \( \Gamma M \)-module \( X \), such that \( ab + c = ab + bc \), for all \( a, b, c \in M \) and \( \alpha, \beta \in \Gamma \),
\[
\sigma^i \tau = \sigma^i \tau, \quad \tau^i = \tau^i, \quad \sigma^i \tau^i = \sigma^i \tau^i = \tau^i \sigma^i \quad \text{and} \quad \sigma^i \tau^i = \tau^i \sigma^i
\]
then \( \theta \) is a generalized Jordan triple \((\sigma, \tau)\)-higher homomorphism.

2. Generalized Jordan \((\sigma, \tau)\)-Higher Homomorphisms of \( \Gamma M \)-Modules

**Definition (2.1):**

Let \( F = (f_i)_{i \in \mathbb{N}} \) be a family of additive mappings of a \( \Gamma \)-ring \( M \) into a left \( \Gamma M \)-module \( X \) and \( \sigma, \tau \) be two endomorphisms of \( M \). \( F \) is called a generalized \((\sigma, \tau)\)-higher homomorphism if there exists a \((\sigma, \tau)\)-higher homomorphism \( \theta = (\phi_i)_{i \in \mathbb{N}} \) from a \( \Gamma \)-ring \( M \) into a left \( \Gamma M \)-module \( X \), such that for all \( a, b \in M, \alpha \in \Gamma \) and \( n \in \mathbb{N} \)
\[
f_n(a \sigma b) = \sum_{i=1}^{n} f_i(\sigma^i(a)) \alpha \phi_i(\tau^i(b))
\]

Where \( \theta \) is called the relating \((\sigma, \tau)\)-higher homomorphism.

**Example (2.2):**

Let \( f = (f_i)_{i \in \mathbb{N}} \) be a generalized \((\sigma, \tau)\)-higher homomorphism of a ring \( R \) into a ring \( R' \). Then there exists a \((\sigma, \tau)\)-higher homomorphism \( \theta = (\phi_i)_{i \in \mathbb{N}} \) from a ring \( R \) into a ring \( R' \), such that for all \( a, b \in R \) and \( n \in \mathbb{N} \)
\[
f_n(a \sigma b) = \sum_{i=1}^{n} f_i(\sigma^i(a)) \theta_i(\tau^i(b)).
\]
Let $M = M_{n \times n}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n \\ 0 \end{pmatrix}, n \in \mathbb{Z} \right\}$. Then $M$ is a $\Gamma$-ring.

Let $F = (F_n)_{n \in \mathbb{N}}$ be a family of additive mappings from a $\Gamma$-ring $M$ into a left $\Gamma M$-module $X$, such that for all $(a, b) \in M$

$F_n((a, b)) = (f_n(a), f_n(b)).$

Then there exists a $(\sigma, \tau)$-higher homomorphism from a $\Gamma$-ring $M$ into a left $\Gamma M$-module $X$

defined by: $\phi_n((a, b)) = (\theta_n(a), \theta_n(b))$, for all $(a, b) \in M$.

Let $\sigma^n_1, \tau^n_1$ be two endomorphisms of $M$, such that

$\sigma^n_1((a, b)) = ((\sigma^n(a), \sigma^n(b))$,

$\tau^n_1((a, b)) = ((\tau^n(a), \tau^n(b)).$

Then $F$ is a generalized $(\sigma, \tau)$-higher homomorphism.

**Definition (2.3):**

Let $F = (F_n)_{n \in \mathbb{N}}$ be a family of additive mappings of a $\Gamma$-ring $M$ into a left $\Gamma M$-module $X$ and $\sigma, \tau$ be two
endomorphisms of $M$. $F$ is called a generalized Jordan $(\sigma, \tau)$-higher homomorphism if there exists a Jordan $(\sigma, \tau)$-higher homomorphism $\theta = (\phi_n)_{n \in \mathbb{N}}$ from a $\Gamma$-ring $M$ into a left $\Gamma M$-module $X$, such that for all $a, b \in M$, $\alpha \in \Gamma$ and $n \in \mathbb{N}$

$f_n((a \alpha b + b \alpha a) = \sum_{i=0}^{n} f_i(\sigma^i(a)) \alpha \phi_i(\tau^i(b)) + \sum_{i=0}^{n} f_i(\sigma^i(b)) \alpha \phi_i(\tau^i(a))$

Where $\theta$ is called the relating Jordan $(\sigma, \tau)$-higher homomorphism.

**Remark (2.4):**

Clearly every generalized $(\sigma, \tau)$-higher homomorphism is a generalized Jordan $(\sigma, \tau)$-higher homomorphism but the converse is not true in general, as shown by the following example.

**Example (2.5):**

Let $S$ be any $\Gamma$-ring with nontrivial involution $\ast$ and $\Gamma$ be the set of all integer numbers, let $M = S \oplus S$, such that $x \in Z(S)$, $s_1 a \alpha s_2 = 0$, for all $s_1, s_2 \in S$, $\alpha \in \Gamma$, $s_1 \neq s_2$ and $a^2 = a$.

Let $F = (F_n)_{n \in \mathbb{N}}$ be a family of additive mappings of a $\Gamma$-ring $M$ into a left $\Gamma M$-module $X$, such that for all $(s, t) \in M$ and $n \in \mathbb{N}$ defined by:

$f_n((s, t)) = \begin{cases} (-2n) \alpha a \alpha s \ast (n-1) t \ast, & n = 1, 2 \\ 0, & n \geq 3 \end{cases}$

for all $(s, t) \in M$.

Then there exists a Jordan $(\sigma, \tau)$-higher homomorphism from a $\Gamma$-ring $M$ into a left $\Gamma M$-module $X$ defined by

$\phi_n((s, t)) = \begin{cases} ((2n) \alpha a \alpha s \ast (n-1) t \ast, & n = 1, 2 \\ 0, & n \geq 3 \end{cases}$

for all $(s, t) \in M$.

Let $\sigma^n, \tau^n$ be two endomorphisms of $M$, such that $\sigma^n((s, t)) = (ns, t)$, $\tau^n((s, t)) = (n^2 s, t)$. Then $F$ is a generalized Jordan $(\sigma, \tau)$-higher homomorphism but not generalized $(\sigma, \tau)$-higher homomorphism.

**Definition (2.6):**

Let $F = (F_n)_{n \in \mathbb{N}}$ be a family of additive mappings of a $\Gamma$-ring $M$ into a left $\Gamma M$-module $X$ and $\sigma, \tau$ be two
endomorphisms of $M$. $F$ is called a generalized Jordan triple $(\sigma, \tau)$-higher homomorphism if there exists
a Jordan triple \((\sigma,\tau)\)-higher homomorphism \(\theta = (\phi_i)_{i\in\mathbb{N}}\) from a \(\Gamma\)-ring \(M\) into a left \(\Gamma M\)-module \(X\), such that for all \(a, b \in M\), \(\alpha, \beta \in \Gamma\) and \(n \in \mathbb{N}\)

\[
f_n(a\, ab\, \beta\, a + a\, b\, b\, \alpha\, a) = \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(a)) + \sum_{i=1}^{n} f_i(\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(b))\alpha\phi_i(\tau^i(a))
\]

Where \(\theta\) is called the relating Jordan triple \((\sigma,\tau)\)-higher homomorphism.

**Definition (2.7):**

Let \(F = (f_i)_{i\in\mathbb{N}}\) be a family of additive mappings of a \(\Gamma\)-ring \(M\) into a left \(\Gamma M\)-module \(X\) and \(\sigma, \tau\) be two endomorphisms of \(M\). \(F\) is called a **generalized \((\sigma,\tau)\)-higher anti homomorphism** if there exists a \((\sigma,\tau)\)-higher anti homomorphism \(\theta = (\phi_i)_{i\in\mathbb{N}}\) from a \(\Gamma\)-ring \(M\) into a left \(\Gamma M\)-module \(X\), such that for all \(a, b \in M\), \(\alpha \in \Gamma\) and \(n \in \mathbb{N}\)

\[
f_n(a\, ab\, \beta\, a) = \sum_{i=1}^{n} f_i(\sigma^i(b))\alpha\phi_i(\tau^i(a))
\]

Where \(\theta\) is called the relating \((\sigma,\tau)\)-higher anti homomorphism.

**Lemma (2.8):**

Let \(F = (f_i)_{i\in\mathbb{N}}\) be a generalized Jordan triple \((\sigma,\tau)\)-higher homomorphism of a \(\Gamma\)-ring \(M\) into a left \(\Gamma M\)-module \(X\), then for all \(a, b, c \in M\), \(\alpha, \beta \in \Gamma\) and for every \(n \in \mathbb{N}\)

(i) If \(\sigma^i = \sigma^j\), \(\tau^i = \tau^j\), \(\sigma^i\tau^i = \sigma^j\tau^j\) and \(\sigma^i = \tau^j\)

\[
f_n(a\, ab\, \beta\, a + c\, ab\, \alpha\, a) = \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(c)) + \sum_{i=1}^{n} f_i(\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(b))\alpha\phi_i(\tau^i(c))
\]

(ii) \(f_n(a\, ab\, \beta\, c + c\, ab\, \beta\, a) = \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(c)) + \sum_{i=1}^{n} f_i(\sigma^i(a))\beta\phi_i(\sigma^i\tau^{n-i}(b))\alpha\phi_i(\tau^i(c))
\]

(iii) In particular, if \(M\) be a commutative \(\Gamma\)-ring and \(X\) is a \(2\)-torsion free \(\Gamma\)-ring, then

\[
f_n(a\, ab\, \beta\, c + c\, ab\, \alpha\, a) = \sum_{i=1}^{n} f_i(\sigma^i(c))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\beta\phi_i(\tau^i(c)) + \sum_{i=1}^{n} f_i(\sigma^i(c))\beta\phi_i(\sigma^i\tau^{n-i}(b))\alpha\phi_i(\tau^i(c))
\]

**Proof:**

(i) Replace \(a\, b + b\, a\) for \(b\) in Definition (2.3), we get:

\[
f_n(a\, a\, (ab\, \beta\, a) + (a\, b\, b\, \alpha\, a)) = \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha\phi_i(\tau^i(ab\, b\, \beta\, a)) + \sum_{i=1}^{n} f_i(\sigma^i(a))\beta\phi_i(\tau^i(ab\, \beta\, a))
\]

\[
= \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha\phi_i(\tau^i(a)\beta\tau^i(b) + \tau^i(b)\beta\tau^i(a)) + \sum_{i=1}^{n} f_i(\sigma^i(a))\beta\phi_i(\tau^i(b) + \tau^i(b)\beta\tau^i(a))\alpha\phi_i(\tau^i(a))
\]

\[
= \sum_{i=1}^{n} f_i(\sigma^i(a))\alpha\left(\sum_{j=1}^{n} \phi_j(\sigma^j\tau^i(a))\beta\phi_j(\tau^j(b)) + \sum_{j=1}^{n} \phi_j(\sigma^j\tau^i(b))\beta\phi_j(\tau^j(a))\right) + \sum_{i=1}^{n} f_i(\sigma^i(a))\beta\phi_i(\tau^i\sigma_i(a)\phi_i(\tau^i(a))
\]

Page 52 ©2016 RS Publication, rspublicationhouse@gmail.com
Since $\sigma^i = \sigma^i$, $\tau^i = \tau^i$, $\sigma^i \tau^i = \sigma^i \tau^{n-1}$ and $\sigma^i \tau^i = \tau^i \sigma^i$

\[
\sum_{i=1}^{\infty} f_i(\sigma^i(a))\alpha \Phi_i(\sigma^i \tau^{n-1}(b))\beta \Phi_i(\tau^i(a)) + \sum_{i=1}^{\infty} f_i(\sigma^i(a))\beta \Phi_i(\sigma^i \tau^{n-1}(b))\alpha \Phi_i(\tau^i(a)) + \sum_{i=1}^{\infty} f_i(\sigma^i(a))\beta \Phi_i(\sigma^i \tau^{n-1}(b))\alpha \Phi_i(\tau^i(a)) + \sum_{i=1}^{\infty} f_i(\sigma^i(b))\beta \Phi_i(\sigma^i \tau^{n-1}(a))\alpha \Phi_i(\tau^i(a))
\]

On the other hand:

\[
f_n(a\alpha(\alpha \beta + \beta \alpha) + (\alpha \beta \beta + \beta \alpha)\alpha a) = n(a\alpha\alpha\beta\beta + a\alpha\beta\alpha + a\beta\beta \alpha \alpha + b \beta \alpha \alpha)
\]

\[
= \sum_{i=1}^{\infty} f_i(\sigma^i(a))\alpha \Phi_i(\sigma^i \tau^{n-1}(a))\beta \Phi_i(\tau^i(a)) + \sum_{i=1}^{\infty} f_i(\sigma^i(a))\beta \Phi_i(\sigma^i \tau^{n-1}(a))\alpha \Phi_i(\tau^i(a)) + \sum_{i=1}^{\infty} f_i(\sigma^i(b))\beta \Phi_i(\sigma^i \tau^{n-1}(a))\alpha \Phi_i(\tau^i(a)) + \sum_{i=1}^{\infty} f_i(\sigma^i(b))\beta \Phi_i(\sigma^i \tau^{n-1}(a))\alpha \Phi_i(\tau^i(a))
\]

\[
\alpha \alpha(\beta \beta)(\alpha \alpha)
\]

\[
\beta \beta(\beta \alpha)
\]

\[
\alpha \alpha \beta \alpha + \alpha \beta \beta \alpha \alpha + b \beta \alpha \alpha
\]

...(1)

(ii) Replace $a + c$ for $a$ in Definition (2.6), we get:

\[
f_n((a + c)\alpha b \beta (a + c)) = \sum_{i=1}^{\infty} f_i(\sigma^i(a + c))\alpha \Phi_i(\sigma^i \tau^{n-1}(b))\beta \Phi_i(\tau^i(a + c))
\]

\[
= \sum_{i=1}^{\infty} f_i(\sigma^i(a))\alpha \Phi_i(\sigma^i \tau^{n-1}(b))\beta \Phi_i(\tau^i(a + c)) + \sum_{i=1}^{\infty} f_i(\sigma^i(a))\alpha \Phi_i(\sigma^i \tau^{n-1}(b))\beta \Phi_i(\tau^i(c)) + \sum_{i=1}^{\infty} f_i(\sigma^i(c))\alpha \Phi_i(\sigma^i \tau^{n-1}(b))\beta \Phi_i(\tau^i(c)) + \sum_{i=1}^{\infty} f_i(\sigma^i(c))\alpha \Phi_i(\sigma^i \tau^{n-1}(b))\beta \Phi_i(\tau^i(c))
\]

\[
\alpha \alpha \beta \alpha + \alpha \beta \beta \alpha \alpha + b \beta \alpha \alpha
\]

...(2)

On the other hand:

\[
f_n(a\alpha b \beta c + c \alpha b \beta \alpha)
\]

\[
= \sum_{i=1}^{\infty} f_i(\sigma^i(a))\alpha \Phi_i(\sigma^i \tau^{n-1}(b))\beta \Phi_i(\tau^i(c)) + \sum_{i=1}^{\infty} f_i(\sigma^i(c))\alpha \Phi_i(\sigma^i \tau^{n-1}(b))\beta \Phi_i(\tau^i(c))
\]

\[
\alpha \alpha \beta \alpha + \alpha \beta \beta \alpha \alpha + b \beta \alpha \alpha
\]

...(1)

(ii) Replace $a + c$ for $a$ in Definition (2.6), we get:

\[
f_n(a\alpha b \beta c + c \alpha b \beta \alpha)
\]

\[
= \sum_{i=1}^{\infty} f_i(\sigma^i(a))\alpha \Phi_i(\sigma^i \tau^{n-1}(b))\beta \Phi_i(\tau^i(c)) + \sum_{i=1}^{\infty} f_i(\sigma^i(c))\alpha \Phi_i(\sigma^i \tau^{n-1}(b))\beta \Phi_i(\tau^i(c))
\]

\[
\alpha \alpha \beta \alpha + \alpha \beta \beta \alpha \alpha + b \beta \alpha \alpha
\]

...(2)

(iii) By (ii) and since $M$ be a commutative $\Gamma$- rings and $X$ is a 2-torsion free, then

\[
f_n(a\alpha b \beta c + c \alpha b \beta \alpha) = \sum_{i=1}^{\infty} f_i(\sigma^i(a))\alpha \Phi_i(\sigma^i \tau^{n-1}(b))\beta \Phi_i(\tau^i(c))
\]

(iv) Replace $a$ for $\beta$ in (ii), we get:

\[
f_n(a\alpha b \alpha c + c \alpha b \alpha a)
\]

\[
= \sum_{i=1}^{\infty} f_i(\sigma^i(a))\alpha \Phi_i(\sigma^i \tau^{n-1}(b))\alpha \Phi_i(\tau^i(c)) + \sum_{i=1}^{\infty} f_i(\sigma^i(c))\alpha \Phi_i(\sigma^i \tau^{n-1}(b))\alpha \Phi_i(\tau^i(a))
\]
Definition (2.9):
Let $F = (f_i)_{i \in \mathbb{N}}$ be a generalized Jordan $(\sigma,\tau)$-higher homomorphism of a $\Gamma$-ring $M$ into a left $\Gamma M$-module $X$, then for all $a, b \in M, \alpha \in \Gamma$ and $n \in \mathbb{N}$, we define:

$$\delta_n(a, b)_\alpha = f_n(a \alpha b) - \sum_{i=1}^{n} f_{i}(\sigma^i(a))\alpha \phi_i(\tau^i(b))$$

Lemma (2.10):
Let $F = (f_i)_{i \in \mathbb{N}}$ be a generalized Jordan $(\sigma,\tau)$-higher homomorphism of a $\Gamma$-ring $M$ into a left $\Gamma M$-module $X$, then for all $a, b, c \in M, \alpha, \beta \in \Gamma$ and $n \in \mathbb{N}$:

(i) $\delta_n(a, b)_\alpha = -\delta_n(b, a)_\alpha$
(ii) $\delta_n(a + b, c)_\alpha = \delta_n(a, c)_\alpha + \delta_n(b, c)_\alpha$
(iii) $\delta_n(a, b + c)_\alpha = \delta_n(a, b)_\alpha + \delta_n(a, c)_\alpha$
(iv) $\delta_n(a, b + \beta)_\alpha = \delta_n(a, b + \beta)_\alpha$

Proof:
(i) By Definition (2.3)
$$f_n(a \alpha b + b \alpha a) = \sum_{i=1}^{n} f_{i}(\sigma^i(a))\alpha \phi_i(\tau^i(b)) + \sum_{i=1}^{n} f_{i}(\sigma^i(b))\alpha \phi_i(\tau^i(a))$$

(ii) $\delta_n(a + b, c)_\alpha = f_n((a + b)\alpha c) - \sum_{i=1}^{n} f_{i}(\sigma^i(a + b))\alpha \phi_i(\tau^i(c))$

(iii) $\delta_n(a, b + c)_\alpha = f_n(a \alpha (b + c)) - \sum_{i=1}^{n} f_{i}(\sigma^i(a))\alpha \phi_i(\tau^i(b + c))$

(iv) $\delta_n(a, b + \beta)_\alpha = f_n(a (\alpha + \beta) b) - \sum_{i=1}^{n} f_{i}(\sigma^i(a))\alpha + \beta \phi_i(\tau^i(b))$

Remark (2.11):
Note that $F = (f_i)_{i \in \mathbb{N}}$ is a generalized $(\sigma,\tau)$-higher homomorphism from a $\Gamma$-ring $M$ into a left $\Gamma M$-module $X$ if and only if $\delta_n(a, b)_\alpha = 0$, for all $a, b \in M, \alpha \in \Gamma$ and $n \in \mathbb{N}$.
Lemma (2.12):

Let $F = (f_{ij})_{i,j \in N}$ be a generalized Jordan $(\sigma, \tau)$-higher homomorphism of a $\Gamma$-ring $M$ into a left $\Gamma M$-module $X$, such that $(\sigma^n)^2 = \sigma^n$, $\tau^n \sigma^n = \sigma^n$, $\sigma^i \tau^{n-i} = \tau^i \sigma^i$ and $\sigma^i \tau^i = \tau^i \sigma^i$, for all $i \in \mathbb{N}$,

then for all $a, b, m \in M$, $\alpha, \beta \in \Gamma$ and $n \in \mathbb{N}$

(i) $\delta_n(\sigma^n(a), \sigma^n(b)) = \alpha \phi_n(\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(a))$

(ii) $\delta_n(\sigma^n(a), \sigma^n(b)) = \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(b), \tau^n(a))$

(iii) $\delta_n(\sigma^n(a), \sigma^n(b)) = \alpha \phi_n(\sigma^n(m)) \beta G_n(\tau^n(a), \tau^n(b))$

Proof:

(i) We prove by the induction, if $n = 1$

Let $w = aab\beta m\beta ba + ba a\beta m\beta aab$, since $F$ is a generalized Jordan $(\sigma, \tau)$-homomorphism

$F(w) = F(aaa)\beta m\beta b a a + b a a\beta m\beta a a b$

$= F(\sigma(a))\omega(\sigma(\tau(b) \beta m\beta b) a a) + F(\sigma(b))\omega(\sigma(\sigma(\beta m\beta a)) a a(\tau(b)))$

$= F(\sigma(a))\omega(\sigma(\sigma(\beta)) \beta(\sigma(\sigma(\tau(\beta))) \omega(\tau(\sigma(a))) a a) + F(\sigma(b))\omega(\sigma(\sigma(\beta)) \omega(\tau(\sigma(a))) a a(\tau(b)))$

$= -F(\sigma(a))\omega(\tau(b)) \beta(\sigma(\sigma(\tau(a))) \omega(\tau(\sigma(a))) a a) + F(\sigma(b))\omega(\sigma(\sigma(\tau(a))) \omega(\tau(\sigma(a))) a a(\tau(b)))$

$= F(\sigma(a))\omega(\tau(b)) \beta(\sigma(\sigma(\tau(a))) \omega(\tau(\sigma(a))) a a) + F(\sigma(b))\omega(\sigma(\sigma(\tau(a))) \omega(\tau(\sigma(a))) a a(\tau(b)))$

$= 0$

On the other hand

$F(w) = F(aaa)\beta m\beta b a a + b a a\beta m\beta a a b$

$= F(\sigma(a))\omega(\sigma(\tau(b) \beta m\beta b) a a) + F(\sigma(b))\omega(\sigma(\sigma(\beta m\beta a)) a a(\tau(b)))$

$= F(\sigma(a))\omega(\sigma(\sigma(\beta)) \beta(\sigma(\sigma(\tau(\beta))) \omega(\tau(\sigma(a))) a a) + F(\sigma(b))\omega(\sigma(\sigma(\beta)) \omega(\tau(\sigma(a))) a a(\tau(b)))$

$= F(\sigma(a))\omega(\tau(b)) \beta(\sigma(\sigma(\tau(a))) \omega(\tau(\sigma(a))) a a) + F(\sigma(b))\omega(\sigma(\sigma(\tau(a))) \omega(\tau(\sigma(a))) a a(\tau(b)))$

$= 0$

Compare (1), (2) and since $\tau^2 = \tau$

$0 = -F(\sigma(aab))\beta(\tau(\sigma(a))) \beta(\tau(\sigma(a))) a a + F(\sigma(aab))\beta(\tau(\sigma(a))) \beta(\tau(\sigma(a))) a a$

$= 0$

since $\sigma^2 = \sigma$ and $\tau^2 = \tau$
Thus, we have:
\[ \delta(\sigma(a), \sigma(b))\beta(\sigma(m))\beta G(\tau(a), \tau(b))a + \delta(\sigma(b), \sigma(a))\beta(\sigma(m))\beta G(\tau(a), \tau(b))a = 0 \]

Now, let us assume that:
\[ \delta_\text{s}(\sigma'(a), \sigma'(b))_\text{s} \beta \phi_\text{s}(\sigma'(m))\beta G_\text{s}(\tau'(b), \tau'(a)) + \delta_\text{s}(\sigma'(b), \sigma'(a))_\text{s} \beta \phi_\text{s}(\sigma'(m))\beta G_\text{s}(\tau'(a), \tau'(b)) = 0 \]

for all \( a, b, m \in M \) and \( s, n \in N, s < n \).

Let \( w = aabm \beta sbaa + baabm \beta saa \)

Since \( F \) is a generalized Jordan \((\sigma, \tau)\)-higher homomorphism, then

\[ f_\text{n}(w) = f_\text{n}(a(a(b(b\beta m b b b)))a + b(a(\beta \beta m b a)ab) \]

\[ = \sum_{i=1}^{n} f_i(\sigma'(a))\alpha \phi_i(\sigma'(\tau(n-1)(b\beta m b b b))a \phi_i(\tau'(a)) + \sum_{i=1}^{n} f_i(\sigma'(b))\alpha \phi_i(\sigma'(\tau(n-1)(a \beta b m a))a \phi_i(\tau'(b)) \]

\[ = \sum_{i=1}^{n} f_i(\sigma'(a))\alpha \phi_i(\sigma'(\tau(n-1)(b)))b \phi_i(\sigma'(\tau(n-1)(a \beta b m a)))b \phi_i(\tau'(a)) + \sum_{i=1}^{n} f_i(\sigma'(b))\alpha \phi_i(\sigma'(\tau(n-1)(a)))b \phi_i(\sigma'(\tau(n-1)(a \beta b m a)))b \phi_i(\tau'(b)) \]

\[ = f_\text{n}(\sigma'(a))\alpha \phi_\text{n}(\sigma'(\tau(n-1)(b)))b \phi_\text{n}(\sigma'(\tau(n-1)(a \beta b m a)))b \sum_{j=1}^{n} \phi_j(\tau'(\tau(n-1)(b)))a \phi_\text{n}(\tau'(a)) + \sum_{j=1}^{n} f_j(\sigma'(a))\alpha \phi_j(\sigma'(\tau(n-1)(b)))b \phi_j(\sigma'(\tau(n-1)(a \beta b m a)))b \sum_{j=1}^{n} \phi_j(\tau'(\tau(n-1)(b)))a \phi_j(\tau'(a)) \]

On the other hand:

\[ \dots (1) \]
\[
\begin{align*}
 f_n(w) &= f_0((aa)\beta)\beta m \beta(baa) + (baa)\beta m \beta(aab) \\
 &= \sum_{n=1}^{\infty} f_n(\sigma' (aa))\beta \phi(\alpha' \tau^{n-1}(m)) \beta \phi(\tau' (baa)) + \\
 &+ \sum_{n=1}^{\infty} f_n(\sigma' (b)\alpha a)\beta \phi(\alpha' \tau^{n-1}(m)) \beta \phi(\tau' (aab)) \\
 &= \sum_{n=1}^{\infty} f_n(\sigma' (aa)\beta)\phi(\tau^{n-1}(m)) \beta( \sum_{i=1}^{n} \phi_i(\sigma' \tau(a))\alpha \phi(\tau'(b)) + \sum_{j=1}^{n} \phi_j(\sigma' \tau(b))\alpha \phi(\tau'(a)) - \\
 &- \phi_i(\tau'(aa)) \bigg) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_n(\sigma' (a))\alpha \phi(\tau'(a)) \beta \phi(\tau'(aa)) + \sum_{j=1}^{\infty} f_n(\sigma' (b))\alpha \phi(\tau'(a)) \beta \phi(\tau'(aa)) - \\
 &- f_n(\sigma' (ab)) \beta \phi(\tau'(aa)) \\
 &= -\sum_{n=1}^{\infty} f_n(\sigma' (aa)\beta)\phi(\alpha' \tau^{n-1}(m)) \beta \phi(\tau'(aa)) - \sum_{n=1}^{\infty} \phi_i(\sigma' \tau(a))\alpha \phi(\tau'(b)) - \\
 &- \sum_{j=1}^{\infty} \phi_j(\sigma' \tau(b))\alpha \phi(\tau'(a)) + \sum_{j=1}^{\infty} f_n(\sigma' (a))\alpha \phi(\tau'(a)) \beta \phi(\tau'(ab)) + \\
 &+ \sum_{j=1}^{\infty} f_n(\sigma' (b))\alpha \phi(\tau'(a)) \beta \phi(\tau'(ab)) \\
 &= -\sum_{n=1}^{\infty} f_n(\sigma' (aa)\beta)\phi(\alpha' \tau^{n-1}(m)) \beta G_i(\tau'(a), \tau'(b)) = \\
 &- \sum_{n=1}^{\infty} f_n(\sigma' (aa)\beta)\phi(\alpha' \tau^{n-1}(m)) \beta G_i(\tau'(b), \tau'(a)) + \\
 &+ \sum_{n=1}^{\infty} f_n(\sigma' (a))\alpha \phi(\tau'(a)) \beta \phi(\tau'(ab)) + \\
 &+ \sum_{n=1}^{\infty} f_n(\sigma' (b))\alpha \phi(\tau'(b)) \beta \phi(\tau'(ab)) \\
 &= -f_n(\sigma' (aa)\beta)\phi(\alpha' \tau^{n-1}(m)) \beta G_i(\tau'(a), \tau'(b)) = \\
 &- f_n(\sigma' (aa)\beta)\phi(\alpha' \tau^{n-1}(m)) \beta G_i(\tau'(b), \tau'(a)) + \\
 &+ \sum_{n=1}^{\infty} f_n(\sigma' (a))\alpha \phi(\tau'(a)) \beta \phi(\tau'(ab)) + \\
 &+ \sum_{n=1}^{\infty} f_n(\sigma' (b))\alpha \phi(\tau'(b)) \beta \phi(\tau'(ab)) \\
 \end{align*}
\]

Compare (1), (2) and since \(\sigma^{n'} = \sigma^n \cdot \tau^n \sigma^n = \sigma^n \cdot \sigma^{n-1} = \tau^i \sigma^i\) and \(\sigma' = \tau^{i'}\), for all \(i \in N\).

\[
O = -f_n(\sigma' (a) \alpha b) \beta \phi(\alpha' \tau^{n-1}(m)) \beta G_i(\tau'(a), \tau'(b)) = \\
- f_n(\sigma' (a) \alpha b) \beta \phi(\alpha' \tau^{n-1}(m)) \beta G_i(\tau'(b), \tau'(a)) + \\
+ \sum_{n=1}^{\infty} f_n(\sigma' (a))\alpha \phi(\tau'(a)) \beta \phi(\tau'(ab)) + \\
+ \sum_{n=1}^{\infty} f_n(\sigma' (b))\alpha \phi(\tau'(b)) \beta \phi(\tau'(ab))
\]

\[\ldots(2)\]
\[
\sum_{i=1}^{n} f_i(\sigma^i) = \sum_{i=1}^{n} f_i(\sigma^i(\sigma^i \tau^{-1}(b))) = \sum_{i=1}^{n} f_i(\sigma^i(\sigma^i \tau^{-1}(a))) = \sum_{i=1}^{n} f_i(\sigma^i(\sigma^i(\tau^{-1}(m)))) = \sum_{i=1}^{n} f_i(\sigma(\sigma^i \tau^{-1}(m)) \beta G_i(\tau^{-1}(a), \tau^{-1}(b)))_a
\]

0 = \sum_{i=1}^{n} f_i(\sigma^i(\sigma(\sigma^i(\tau^{-1}(m)) \beta G_i(\tau^{-1}(a), \tau^{-1}(b))))_a

\]

By our hypothesis, we have:
\[
\delta_n(\sigma^i(\tau^{-1}(m)) \beta G_i(\tau^{-1}(a), \tau^{-1}(b)))_a
= \delta_n(\sigma^i(\tau^{-1}(m)) \beta G_i(\tau^{-1}(a), \tau^{-1}(b)))_a
= \delta_n(\sigma^i(\tau^{-1}(m)) \beta G_i(\tau^{-1}(a), \tau^{-1}(b)))_a = 0.
\]

(ii) Replace \( \beta \) by \( \alpha \) in (i), we get (ii).

(iii) Interchanging \( \alpha \) and \( \beta \) in (i), we get (iii).

**Lemma (2.13):**

Let \( F = (f_i)_{i \in \mathbb{N}} \) be a generalized Jordan (\( \sigma, \tau \)) - higher homomorphism from a \( \Gamma \)-ring \( M \) into a 2- torision free prime left \( \Gamma \)M - module \( X \), then for all \( a, b, m \in M, \alpha, \beta \in \Gamma \) and \( n \in \mathbb{N} \)

(i) \[\delta_n(\sigma^i(a), \sigma^i(b))_a \beta g_n(\sigma^i(m)) \beta G_n(\tau^n(a), \tau^n(b)) = 0\]

(ii) \[\delta_n(\sigma^i(a), \sigma^i(b))_a \alpha g_n(\sigma^i(m)) \alpha G_n(\tau^n(a), \tau^n(b)) = 0\]
(iii) \( \delta_n(\sigma^n(a), \sigma^n(b))_\beta \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(a), \tau^n(b))_\beta = \delta_n(\sigma^n(b), \sigma^n(a))_\alpha \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(b), \tau^n(a))_\alpha = 0 \)

**Proof:**

(i) By Lemma (2.12) (i), we have:
\[
\delta_n(\sigma^n(a),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(a),\tau^n(b))_\alpha + \\
\delta_n(\sigma^n(b),\sigma^n(a))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(b),\tau^n(a))_\alpha = 0
\]

And by Lemma (Let X be a 2-torsion free semiprime \( \Gamma-M \)-module X and suppose that \( a, b \in \Gamma-M \)-module X if \( a \Gamma b + b \Gamma a = 0 \) for all \( x \in X \), then \( a \Gamma b = b \Gamma a = 0 \), we get:
\[
\delta_n(\sigma^n(a),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(a),\tau^n(b))_\alpha = 0
\]

(ii) Replace \( \alpha \) for \( \beta \) in (i), we obtain (ii).

(iii) Interchanging \( \alpha \) and \( \beta \) in (i), we obtain (iii).

**Lemma (2.14):**

Let \( F = (f_i)_{i \in N} \) be a generalized Jordan (\( \sigma,\tau \))-higher homomorphism from a \( \Gamma \)-ring \( M \) into a prime \( \Gamma \)-module \( X \), then for all \( a, b, c, d, m \in M, \alpha, \beta \in \Gamma \) and \( n \in N \)

(i) \( \delta_n(\sigma^n(a),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(a)+\tau^n(b))_\alpha = 0 \)

(ii) \( \delta_n(\sigma^n(a),\sigma^n(b))_\alpha \alpha \phi(a(\sigma^n(m)))\alpha G_n(\tau^n(a)+\tau^n(b))_\alpha = 0 \)

(iii) \( \delta_n(\sigma^n(a),\sigma^n(b))_\beta \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(a)+\tau^n(b))_\alpha = 0 \)

**Proof:**

(i) Replacing \( a + c \) for \( a \) in Lemma (2.13) (i), we get:
\[
\delta_n(\sigma^n(a+c),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(a+c),\tau^n(b))_\alpha = 0 \\
\delta_n(\sigma^n(a),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(a),\tau^n(b))_\alpha + \\
\delta_n(\sigma^n(a),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(a),\tau^n(b))_\alpha + \\
\delta_n(\sigma^n(c),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(c),\tau^n(b))_\alpha + \\
\delta_n(\sigma^n(c),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(c),\tau^n(b))_\alpha = 0 \\
\delta_n(\sigma^n(a),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(a),\tau^n(b))_\alpha + \\
\delta_n(\sigma^n(a),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(a),\tau^n(b))_\alpha = 0
\]

By Lemma (2.13) (i), we get:
\[
\delta_n(\sigma^n(a),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(b),\tau^n(a))_\alpha = 0 \\
\delta_n(\sigma^n(c),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(b),\tau^n(a))_\alpha = 0 \\
\beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(b),\tau^n(a))_\alpha = 0
\]

Therefore, we get:
\[
\delta_n(\sigma^n(a),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(b),\tau^n(c))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(b),\tau^n(c))_\alpha = 0 \\
\delta_n(\sigma^n(a),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(b),\tau^n(c))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(b),\tau^n(c))_\alpha = 0 \\
\delta_n(\sigma^n(c),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(b),\tau^n(c))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(b),\tau^n(c))_\alpha = 0 \\
\delta_n(\sigma^n(a),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(b),\tau^n(c))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(b),\tau^n(c))_\alpha = 0
\]

Since \( X \) is a prime \( \Gamma \)-module and therefore:
\[
\delta_n(\sigma^n(a),\sigma^n(b))_\alpha \beta \phi(a(\sigma^n(m)))\beta G_n(\tau^n(b),\tau^n(c))_\alpha = 0 \quad \text{... (1)}
\]

Replacing \( b + d \) for \( b \) in Lemma (2.13) (i), we get:
\[ \delta_n(\sigma^a, \sigma^b(\tau + x) )_{\alpha} \beta A_n(\sigma^a) \beta G_n(\tau^p), \tau^p(a)) = 0 \]

\[ \delta_n(\alpha^a, \sigma^b(\tau + x) )_{\alpha} \beta A_n(\sigma^a) \beta G_n(\tau^p), \tau^p(a)) = 0 \]

By Lemma (2.13) (i), we get:

\[ \delta_n(\alpha^a, \sigma^b(\tau + x) )_{\alpha} \beta A_n(\sigma^a) \beta G_n(\tau^p), \tau^p(a)) = 0 \]

Therefore, we get:

\[ \delta_n(\alpha^a, \sigma^b(\tau + x) )_{\alpha} \beta A_n(\sigma^a) \beta G_n(\tau^p), \tau^p(a)) = 0 \]

Since X is a prime \( \Gamma \)-module and therefore:

\[ \delta_n(\alpha^a, \sigma^b(\tau + x) )_{\alpha} \beta A_n(\sigma^a) \beta G_n(\tau^p), \tau^p(a)) = 0 \]

Now, let \( \delta_n(\alpha^a, \sigma^b(\tau + x) )_{\alpha} \beta A_n(\sigma^a) \beta G_n(\tau^p), \tau^p(a)) = 0 \)

By Lemma (2.13) (i) and (1), (2), we get:

\[ \delta_n(\alpha^a, \sigma^b(\tau + x) )_{\alpha} \beta A_n(\sigma^a) \beta G_n(\tau^p), \tau^p(a)) = 0 \]

(ii) Replace \( \alpha \) for \( \beta \) in (i), we obtain (ii).

(iii) Replacing \( \alpha + \beta \) for \( \alpha \) in (ii), we get:

\[ \delta_n(\alpha^a, \sigma^b(\tau + x) )_{\alpha + \beta} \beta A_n(\sigma^a) \beta G_n(\tau^p), \tau^p(a)) = 0 \]

By (i) and (ii), we get:

\[ \delta_n(\alpha^a, \sigma^b(\tau + x) )_{\alpha} \beta A_n(\sigma^a) \beta G_n(\tau^p), \tau^p(a)) = 0 \]

Therefore, we have:

\[ \delta_n(\alpha^a, \sigma^b(\tau + x) )_{\alpha} \beta A_n(\sigma^a) \beta G_n(\tau^p), \tau^p(a)) = 0 \]

Since X is a prime \( \Gamma \)-module, then:

\[ \delta_n(\alpha^a, \sigma^b(\tau + x) )_{\alpha} \beta A_n(\sigma^a) \beta G_n(\tau^p), \tau^p(a)) = 0 \]
3. The Main Result

**Theorem (3.1):**

Every generalized Jordan \((\sigma, \tau)-\)higher homomorphism from a \(\Gamma\)-ring \(M\) into a prime left \(\Gamma M\) - module \(X\) is either generalized \((\sigma, \tau)-\)higher homomorphism or \((\sigma, \tau)-\)higher anti homomorphism.

**Proof:**

Let \(F = (f_i)_{i \in \mathbb{N}}\) be a generalized Jordan \((\sigma, \tau)-\)higher homomorphism of a \(\Gamma\)-ring \(M\) into a prime left \(\Gamma M\) - module \(X\), then by Lemma (2.14)(i):

\[
\delta_n(\sigma^n(a), \sigma^n(b)) \circ_\beta \Phi_n(\sigma^n(m)) \circ \beta G_n(\tau^n(d), \tau^n(c)) \alpha = 0.
\]

Since \(X\) is a prime left \(\Gamma M\) - module therefore either \(\delta_n(\sigma^n(a), \sigma^n(b)) \alpha = 0\) or \(G_n(\tau^n(d), \tau^n(c)) \alpha = 0\), for all \(a, b, c, d \in M, \alpha, \beta \in \Gamma\) and \(n \in \mathbb{N}\).

If \(G_n(\tau^n(d), \tau^n(c)) \alpha \neq 0\), for all \(c, d \in M, \alpha \in \Gamma\) and \(n \in \mathbb{N}\) then for all \(c, d \in M, \alpha \in \Gamma\) and \(n \in \mathbb{N}\) then \(\delta_n(\sigma^n(a), \sigma^n(b)) \alpha = 0\). Hence, we get \(F\) is a generalized \((\sigma, \tau)-\)higher homomorphism.

But if \(G_n(\tau^n(d), \tau^n(c)) \alpha = 0\), for all \(c, d \in M, \alpha \in \Gamma\) and \(n \in \mathbb{N}\) then we get \(F\) is a \((\sigma, \tau)-\)higher anti homomorphism.

**Proposition (3.2):**

Let \(F = (f_i)_{i \in \mathbb{N}}\) be a generalized Jordan \((\sigma, \tau)-\)higher homomorphism from a \(\Gamma\)-ring \(M\) into a 2-torsion free left \(\Gamma M\) - module \(X\), such that \(ab\beta a = a\beta b a\), for all \(a, b \in M\) and \(\alpha, \beta \in \Gamma\), \(a'ab'\beta a' = a'\beta b'a'a'\), for all \(a', b' \in M\) and \(\alpha, \beta \in \Gamma\), \(\sigma_1 = \sigma^1, \tau_1 = \tau^1, \sigma_1^1 = \sigma^1_1 = \sigma^{1-1}, \tau_1^1 = \tau^1_1 = \tau^{1-1}\) and \(\sigma_1^1 = \tau_1^1\) for all \(i \in \mathbb{N}\), then \(F\) is a generalized Jordan triple \((\sigma, \tau)-\)higher homomorphism.

**Proof:**

Replace \(a'\beta b + b\alpha a\) for \(b\) in Definition (2.3), we get :

\[
f_n(\alpha \alpha (a\beta b + b\beta a) + (a\beta b + b\beta a) \alpha a) = \sum_{i=1}^{n} f_i(\sigma^i(\alpha) \alpha \Phi_i(\tau^i(a)) + \sum_{i=1}^{n} f_i(\sigma^i(\alpha) \alpha \Phi_i(\tau^i(a))) \alpha \Phi_i(\tau^i(a))
\]

\[
= \sum_{i=1}^{n} f_i(\sigma^i(\alpha) \alpha \Phi_i(\tau^i(a))) \alpha (\sum_{j=1}^{n} \phi_j(\sigma^j \tau^j(\alpha)) \beta \Phi_j(\tau^j(b)) + \sum_{j=1}^{n} \phi_j(\sigma^j \tau^j(b)) \beta \Phi_j(\tau^j(\alpha))) + \sum_{j=1}^{n} f_i(\sigma^j(\alpha) \beta \Phi_j(\tau^j(a)) \beta \Phi_j(\tau^j(\alpha))) \alpha \Phi_i(\tau^i(a))
\]

Since \(a'\beta b' a' = a'\beta b'a'a'\), for all \(a', b' \in X\) and \(\alpha, \beta \in \Gamma\), \(\sigma_1 = \sigma^1, \tau_1 = \tau^1, \sigma_1^1 = \sigma^1_1 = \sigma^{1-1}, \tau_1^1 = \tau^1_1 = \tau^{1-1}\) and \(\sigma_1^1 = \tau_1^1\) for all \(i \in \mathbb{N}\)

\[
= \sum_{i=1}^{n} f_i(\sigma^i(\alpha) \alpha \Phi_i(\sigma^i \tau^i(a)) \beta \Phi_i(\tau^i(b)) + \sum_{j=1}^{n} f_i(\sigma^j(\alpha) \alpha \Phi_i(\sigma^j \tau^j(\alpha)) \beta \Phi_i(\tau^j(\alpha)) \alpha \Phi_i(\tau^i(a)))
\]

On the other hand:

\[
f_n(\alpha \alpha (a\beta b + b\beta a) + (a\beta b + b\beta a) \alpha a) = f_n(\alpha \alpha \beta b + a \alpha \beta b + a \beta b \alpha a + b \beta a)\alpha a
\]

Since \(ab\beta a = a\beta b a\), for all \(a, b \in M\) and \(\alpha, \beta \in \Gamma\)
\[
\sum_{i=1}^{n} f_i(\alpha(\sigma(a))\alpha\phi_i(\tau^{-1}(\sigma)\beta\phi_i(\tau(b))) + \sum_{i=1}^{n} f_i(\sigma(\tau^i(a))\beta\phi_i(\tau^{-1}(\sigma)\alpha\phi_i(\tau^i(a)) + 2f(a\alpha b\beta a)
\]

Compare (1) and (2), we get:

\[
2f_e(a\alpha b\beta a) = 2\sum_{i=1}^{n} f_i(\sigma^i(a))\alpha\phi_i(\tau^{-1}(b))\beta\phi_i(\tau^i(a)).
\]

Since X is a 2-torsion free, we obtain F is a generalized Jordan triple (\alpha,\beta)-higher homomorphism.

References