

## الأغلفة الاغمارية للمقاسات التبولوجية

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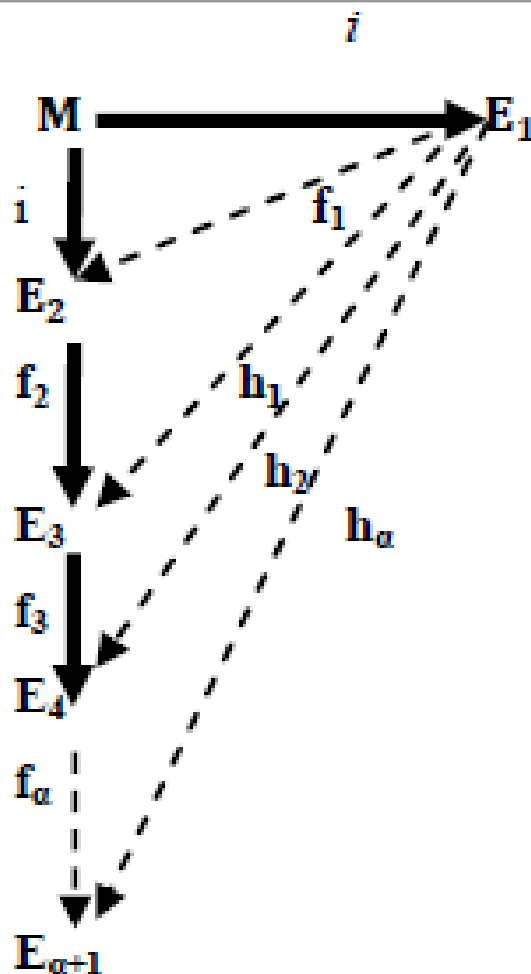
قسم الرياضيات ، كلية التربية ، الجامعة المستنصرية

### المستخلص

ان اهتمامنا الرئيسي هو دراسة الخصائص المقاسية الجبرية تبولوجياً وخصوصاً المقاسات التبولوجية الاغمارية. حصلنا على نتائج عديدة في البحث. بينا كيفية إيجاد الغلاف الاغماري للمقاس التبولوجي من مقاسات تبولوجية أخرى معطاة. أوردنا بمبرهنة الشروط التي تجعل المقاس التبولوجي غلافاً اغمارياً لنفسه.

أوضحنا اذا وجد عدة اغلفة منتهية  $E_1, E_2, \dots, E_n$  لمقاس تبولوجي واحد  $M$  فانهم متكافئين تبولوجياً مقاسياً  $E_\alpha \cong E_{\alpha+1}$

$f_{\alpha+1} \circ f_\alpha = h_\alpha$  حيث  $(\alpha = 1, 2, \dots, n-1)$



## Injective Envelopes of Topological Modules

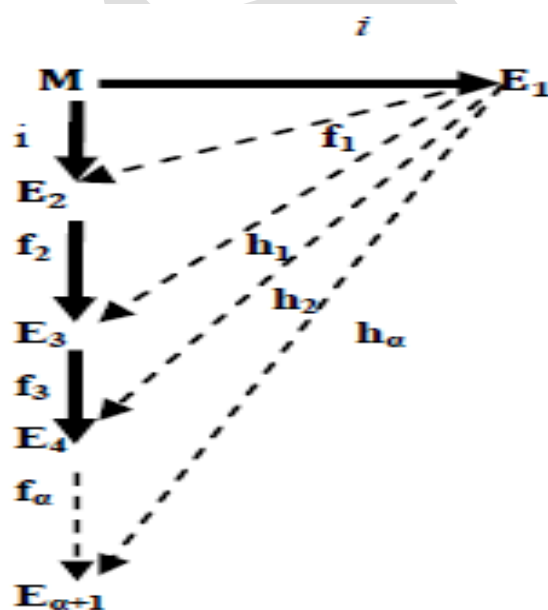
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### Abstract

Our principle aim is to study an algebraic module properties topologically and specially injective topological modules. Among the results in paper are obtained. We show how you can find injective envelope of topological module by other given topological module. We prove the condition to give every injective topological module be injective envelope of itself. We explain if we find a number of injective envelopes  $E_1, E_2, \dots, E_n$  of topological module  $M$  then they are equivalence topological module  $E_\alpha \cong E_{\alpha+1}$  ( $\alpha = 1, 2, \dots, n-1$ ).

Where  $f_{\alpha+1} \circ f_\alpha = h_\alpha$ .



**Key Words:** Injective envelope, injective topological module, homomorphism topological module, essential extension, maximal essential extension.

## Introduction

The aim of this work is to lesson attributes of topological algebra modules [1]. We start with simple cases of algebra and topology, but they are very important, so in our study we concerned with injective envelope of topological module, essential extension of topological module.

In the ending of twentieth century they begin to concern with study of topological groups and at the ending of the 40<sup>th</sup> they concern with lesson topological ring by scientist Cabaskee that used quotient ring as a basic aim and the type of topological metric and his researcher were continued and in his forth research in 1955, he gave the definition of topological modules and the partial topological measurement, the number of research like Dikran Dikranjan, Albertto Tonolo and Nilson touch on the conception of topological modules [3], and they concerned with metric space, they did not limit any specific research about topological modules [5] and they mentioned that in the researcher that study of topological linear space, topological modules and linear modules space. They fundamental neighborhood systems of zero is partial modules or contains partial modules.

In this paper we lesson topological modules especially injective topological modules [4]. The important point of this paper is to evaluate injective envelope of topological modules, and essential extension of topological modules. We obtained some results of injective envelopes of topological modules.

## 1- Topological Modules

In this section we give the fundamental concepts of this work.

### **Definition 1.1** : [4]

A topological group is a set  $G$  together with two structures:

- (1)  $G$  is a group.
- (2) Topology  $T$  on a set  $G$

The two structures are compatible, i.e. the group (binary) operation  $\mu: G \times G \longrightarrow G$ , and the inversion law  $\nu: G \longrightarrow G$  are both continuous maps.

### **Definition 1.2** : [3]

Every group is a topological group with discrete topology  $G = \mathbb{R}$  on the usual topology is topological group.

**Definition 1.3** : [2]

The topological ring  $R$  is a non-empty set together with two structures algebra ring and topology on  $R$  satisfy the following:

1. A map  $(x,y) \longrightarrow x + y$  from  $R \times R \longrightarrow R$  be continuous.
2. A map  $x \longrightarrow -x$  from  $R \longrightarrow R$  be continuous.
3. A map  $(x,y) \longrightarrow xy$  from  $R \times R \longrightarrow R$  be continuous.

**Example (1.4)**:

The discrete topology on the ring  $R$  is a topological ring.

The usual topology on the ring  $R$  is a topological ring.

**Definition (1.5)**: [5]

Let  $R$  be a topological ring. The set  $E$  is called left topological module on  $R$ , if:

1.  $E$  is left module on  $R$ .
2.  $E$  is a topological group.
3. A map  $(\lambda,x) \longrightarrow \lambda x$  from  $R \times E \longrightarrow E$  is continuous for all  $x \in E, \lambda \in R$ .

On the same way, we may define the right topological module.

**Example (1.6)**: [6]

1. The module on a ring from topological module with discrete topology.
2. Every abelian group is topological module on the discrete ring  $\mathbb{Z}$ .

**Definition (1.7)**: [5]

Let  $E$  and  $E'$  be two topological modules on the topological ring  $R, f: E \longrightarrow E'$  is called topological module homomorphism, if:

1.  $f$  is module homomorphism.
2.  $f$  is continuous.

**Definition (1.8):** [4]

Let  $E$  be a topological module on  $R$  (topological ring), the subset  $M$  of  $E$  is a topological submodule of  $E$  if;

1.  $M$  is a submodule of  $E$ .
2.  $M$  is a topological subring of  $E$ .
3. A map  $(\lambda, x) \longrightarrow \lambda x$  from  $R \times M \longrightarrow M$  is continuous.

**Proposition (1.9):** [3]

If  $E$  be topological module of ring  $R$  then every submodule contains zero neighborhood be open.

## 2- Injective Topological Module

**Definition (2.1):** [5]

Let  $E$  left topological module of topological ring  $R$  then  $E$  is injective if  $M'$  be left open submodule of left topological module  $M$  and  $f : M' \longrightarrow E$  be homomorphism topological module then  $f$  be extension of homomorphism topological module from  $M$  to  $E$ .

**Notation (2.2):**

$\text{Ker } f$  is topological submodule of  $E$ , where  $f$  is homomorphism topological module from  $E$  into  $E'$ .

**Proposition (2.3):** [1]

Let  $E$  be discrete topological module,  $E$  is injective iff for any submodule  $N$  from topological module  $E'$ , the homomorphism topological module  $f : N \longrightarrow E$  extension of  $E'$ .

**Proposition (2.4):** [1]

Let  $R$  be topological ring and  $\{E_\alpha\}_{1 \leq \alpha \leq n}$  finite family of injective discrete topological module of  $R$  then the direct sum  $\bigoplus_{1 \leq \alpha \leq n} E_\alpha$  be discrete injective topological module.

**Corollary (2.5):** [2]

Let  $R$  be a topological ring,  $\{E_\alpha\}_{\alpha \in \Delta}$  be a family of injective discrete topological module then the direct sum  $\bigoplus_{\alpha \in \Delta} E_\alpha$  be injective discrete topological module.

**Definition (2.6):** [1]

Let  $E$  be topological module and  $M$  be submodule of  $E$  then  $E$  is injective extension if  $E$  be injective and  $M$  be open of  $E$ .

**Definition (2.7):** [1]

Let  $E$  be topological module of topological ring  $R$  and  $M$  be submodule of  $E$ , then  $M$  is called essential submodule iff  $M$  has non zero intersection from all submodule of  $E$ .

**Definition (2.8):** [3]

A topological module  $E$  is called essential extension of submodule  $M$  if  $M$  be open of  $E$  and any intersection  $M$  of all nonzero submodule from  $E$  is a non zero.

**Proposition (2.9):** [4]

If  $M_1 \subset M_2 \subset M_3$  where  $M_1, M_2, M_3$  are submodules of  $E$  and  $M_2$  be essential extension of  $M_1$ ,  $M_3$  be essential extension of  $M_2$  then  $M_3$  be essential extension of  $M_1$ .

**Proposition (2.10):** [4]

If  $M_1 \subset M_2 \subset M_3 \subset \dots M_n$  where  $M_1, M_2, M_3, \dots, M_n$  ( $n \in \mathbb{N}$ ) are submodules of  $E$  and  $M_{i+1}$  be essential extension of  $M_i$ , then  $M_n$  be essential extension of  $M_1$ , where  $i = 1, 2, \dots, n$ .

**Proof:**

By proposition (2.9)  $M_3$  be essential extension of  $M_1$ , since  $M_2$  be essential extension of  $M_1$  and  $M_3$  be essential extension of  $M_2$ .

On the same way  $M_n$  be essential extension of  $M_1$ .

**Proposition (2.11):** [4]

If  $E_1 \oplus E_2$  be left topological module and injective and  $E_1 \oplus E_2$  be left open submodule from  $F$  then  $F$  be the direct sum of  $E_1 \oplus E_2$  and  $E_1 \oplus E_2$  discrete submodule of  $F$ .

**Proposition (2.12):** [2]

If  $\bigoplus_{1 \leq \alpha \leq n} E_\alpha$  be left topological module and injective,  $\bigoplus_{1 \leq \alpha \leq n} E_\alpha$  be left open submodule from F. Then F direct sum of  $\bigoplus_{1 \leq \alpha \leq n} E_\alpha$  and  $\bigoplus_{1 \leq \alpha \leq n} E_\alpha$  discrete submodule of F.

**Proposition (2.13):** [2]

If  $\bigoplus \{E_\alpha\}_{\alpha \in \Delta}$  be left injective topological module and  $\bigoplus \{E_\alpha\}_{\alpha \in \Delta}$  be left open submodule of F then F be direct sum of  $\bigoplus \{E_\alpha\}_{\alpha \in \Delta}$  and  $\bigoplus \{E_\alpha\}_{\alpha \in \Delta}$  discrete submodule of F.

**Proposition (2.14):**

If  $M_1$  be a left topological module if  $M_2$  be essential extension of  $M_1$  then  $M_1$  be injective.

**Proof:**

Let  $M_1 \neq M_2$ ,  $M_1$  is not essential extension, let  $M'_2$  be injective topological module contains  $M_1$  where  $M_1$  be open submodule of  $M'_2$  and let M be another submodule of  $M_2$  satisfy  $M \cap M_1 = 0$ . Clear,  $M + M_1 / M \subseteq M'_2 / M$  special case  $M + M_1 / M$  be equivalent topological module of  $M_1$ . Claim  $M'_2 / M$  be essential extension of  $M + M_1 / M$ . Let K be submodule of  $M'_2$  contains  $M_1$ , thus  $K/M$  be submodule of  $M'_2 / M$ .

$$\text{Let } M + (M_1 / M) \cap K / M = 0$$

$$K \cap (M_1 + M) \subset M_1$$

$$K \cap M_1 \subset M \cap M_1 = 0$$

Thus  $K \cap M_1 = 0$ , but M is maximal, thus  $K = M$  and  $K/M = 0$ .

So  $M'_2 / M$  be essential extension of  $M_1 \cong M + M_1 / M$ . By hypothesis  $M_1$  is not essential extension of  $M + M_1 / M$ , thus  $M'_2 / M \cong M_1 \Rightarrow M'_2 = M + M_1$ . Thus  $M_1$  be injective by proposition (2.10).

**Corollary (2.15):**

Let  $M_1 \subset M_2 \subset \dots M_n$ , ( $n \in \mathbb{N}$ ),  $M_1, M_2, \dots, M_n$  are be left topological module if  $M_n$  be essential extension of  $M_1$  then  $M_1$  be injective

**Proof:**

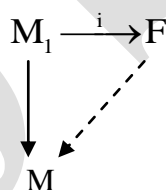
Let  $M_1 \subset M_2 \subset \dots M_n$ , ( $n \in \mathbb{N}$ ) and  $M_n$  be essential extension of  $M$  thus  $M_1$  be injective by theorem (2.14).

**Proposition (2.16):**

If  $M_1 \subset M_2$ ,  $M_2$  be injective topological module and let  $M_1$  be open topological submodule of  $M_2$  be not contain of another submodule and  $M_2$  be essential extension of  $M_1$  then  $M_1$  be injective topological module.

**Proof:**

Let  $F$  be essential extension of  $M$  and  $M_2$  be injective, there exists injective map  $f: F \longrightarrow M_2$ , but  $f(F)$  be submodule of  $M_2$  contain  $M_1$  and essential extension of  $M_1$  thus  $f(F) = M_1$  but  $f$  is one to one, so  $F = M_1$ .



### 3- Injective Envelope

**Definition (3.1):** [3]

Let  $E$  is topological module and  $M$  is submodule of  $E$ . If  $E$  is an injective and essential extension of  $M$  then  $E$  is called injective envelope of  $M$ .

**Examples (3.2):**

(1) Let  $Q$  (group of rational numbers) is discrete module of discrete ring  $Z$  easy we say  $Q$  is an injective topological module, since  $Q$  is discrete thus  $Z$  be open submodule of  $Q$ , we clear that  $Q$  be essential extension of  $Z$ . Let  $M$  is submodule of  $Q \neq \{0\}$ , there exists



$r/s \in Q$ ,  $r, s \in Z$  and  $r/s \neq 0$  thus  $r \neq 0$   $r = s(r/s) \in M$  and  $M \cap Z \neq 0$ . We obtain  $Q$  be essential extension of  $Z$ . Finally  $Q$  be injective envelope of  $Z$ .

(2) Let  $R$  (group of real numbers) is discrete module of discrete ring  $Z$  thus  $R$  is not an injective envelope of  $Z$  as module itself because  $R$  is not an essential extension of  $Z$ .

**Proposition (3.3):** [4]

If  $E$  is an injective topological module of  $M$  and  $M$  is an essential extension of  $E$  then  $M = E$  that is mean  $E$  has not maximal essential extension.

**Proposition (3.4):**

If  $E$  is an injective envelope of  $M$  then  $E$  is maximal essential extension of  $M$ .

**Proof:**

Let  $E$  is an injective envelope of  $M$  by definition (3.1) thus  $E$  be essential extension of  $M$ , but  $E$  have not actual essential extension by proposition (3.3) thus  $E$  be maximal essential extension of  $M$ .

**Proposition (3.5):**

If  $E$  is an injective envelope of submodule  $M$  then  $E$  is minimal essential extension of  $M$ .

**Proof:**

Let  $E$  is an injective envelope of submodule  $M$  by definition (3.1)  $E$  is an injective and  $M$  is an open of  $E$ . Let  $E'$  is submodule of  $E$  and an injective extension of  $M$ , thus  $E'$  is an open of  $E$  from proposition (1.9)  $E$  be essential extension of  $E'$ . But  $E'$  is an injective of  $E$ , thus  $E = E'$ . We obtain  $E$  is minimal injective extension of  $M$ .

**Proposition (3.6):**

$E$  is an injective iff  $E$  is an injective envelope itself.

**Proof:**

( $\Rightarrow$ ) Let  $E$  is an injective

Let  $M$  be submodule of  $E$  by definition (3.1)  $E$  is an essential extension of  $M$ , so  $E$  is maximal essential extension of  $M$  by proposition (3.3) and  $E$  is minimal essential extension of  $M$  by proposition (3.4) thus  $E = M$ .

( $\Leftarrow$ )  $E$  is an injective envelope of itself

Thus  $E$  is an injective topological module and essential extension of  $E$ , we obtain  $E$  is an injective.

**Proposition (3.7):**

If  $E_1$ ,  $E_2$  and  $E_3$  are injective envelopes of topological module  $M$  such that the homeomorphism topological modules  $f : E_1 \longrightarrow E_2$  and  $g : E_2 \longrightarrow E_3$  then there exists homeomorphism topological module  $g \circ f : E_1 \longrightarrow E_3$ .

**Proof:**

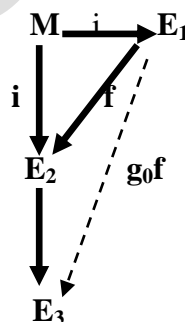
There exist injective map  $g \circ f : E_1 \longrightarrow E_3$ ,

Thus  $g \circ f(E_1)$  is an injective extension of  $M \subset E_3$ .

And  $E_3$  is a minimal injective extension of  $M$  by proposition (3.4).

Thus  $(g \circ f)(E_1) = g(f(E_1))$

$$= g(E_2) = E_3$$



And  $(g \circ f)^{-1}$  is continuous since  $f^{-1}$  and  $g^{-1}$  are continuous. Thus  $g \circ f$  is homeomorphism topological module.

**Corollary (3.8):**

If  $E_1, E_2, E_3, \dots, E_n$  are injective envelopes of topological module  $M$  such that the homeomorphism topological module  $M$   $f_i : E_i \longrightarrow E_{i+1}, i = 1, 2, \dots, n-1$ . Then there exists homeomorphism topological module  $f_n \circ f_1 : E_1 \longrightarrow E_n$ .

**Proof:**

By proposition (3.6) there exist homeomorphism topological modules

$$f_1 : E_1 \longrightarrow E_2$$

$$f_2 : E_2 \longrightarrow E_3$$

$$f_3 : E_3 \longrightarrow E_4$$

$\vdots$

$$f_{n-1} : E_{n-1} \longrightarrow E_n$$

$f_{n-1} \circ \dots \circ f_2 \circ f_1 : E_1 \longrightarrow E_n$ , where  $f_{n-1} \circ \dots \circ f_2 \circ f_1 = h$  and  $f_i^{-1}$  continuous, thus  $h$  is homeomorphism topological module.

**Corollary (3.9):**

If  $E$  is an injective envelope of submodule  $M$  then  $E$  is an injective envelope of an open submodule contains  $M$ .

**Proof:**

Suppose  $M'$  is an open submodule of  $E$  contains  $M$  and  $E$  is not an essential extension of  $M'$ , we obtain  $E$  is not an essential extension of  $M$  that contradiction.

**Proposition (3.10):**

If  $E_1 \oplus E_2$  be left topological module of topological ring  $R$  then  $E_1 \oplus E_2$  have injective envelope.

**Proof:**

Algebraically, we can inject  $E_1 \oplus E_2$  of injective module  $X_1 \oplus X_2$  and there exists topology  $F$  that extension of topological module  $E_1 \oplus E_2$ , we want to prove  $F$  is injective, where  $F = F_1 \oplus F_2$ .

Let  $I$  be open left ideal of  $R$ , and  $f : I \longrightarrow R$  homeomorphism topological module, from injectively  $X_1 \oplus X_2$  and algebraically there exists  $x_1 \oplus x_2 \in X_1 \oplus X_2$  such that  $f(a) = a(x_1 \oplus x_2)$  for all  $a \in I$ .

To show  $x_1 \oplus x_2 \in F = F_1 \oplus F_2$ .

Let  $U$  be neighborhood of zero of  $F$ , since the map  $f$  is continuous that  $V = f^{-1}(U)$  neighborhood of zero of  $I$ , open of  $R$  that  $V$  is neighborhood of zero in  $R$ . By definition of  $f$  thus  $f(V) = V(x_1 \oplus x_2)$  but  $f(V) = U$ , thus  $V(x_1 \oplus x_2) \subset U$  that means  $x_1 \oplus x_2 \in F_1 \oplus F_2 = F$ .

**Proposition (3.11):**

If  $E_1 \oplus E_2 \oplus \dots \oplus E_n$  be left topological module of topological ring  $R$  then  $E_1 \oplus E_2 \oplus \dots \oplus E_n$  have injective envelope.

**Proof:**

Algebraically, we can inject  $E_1 \oplus E_2 \oplus \dots \oplus E_n$  of injective module  $X_1 \oplus X_2 \oplus \dots \oplus X_n$  and there exists topology  $F$  that extension of topological module  $E_1 \oplus E_2 \oplus \dots \oplus E_n$ , we want to prove  $F$  is injective.

Let  $I$  be open left ideal of  $R$ , and  $f : I \longrightarrow R$  homeomorphism topological module, from injectively  $X_1 \oplus X_2 \oplus \dots \oplus X_n$  and algebraically there exists  $x_1 \oplus x_2 \oplus \dots \oplus x_n \in X_1 \oplus X_2 \oplus \dots \oplus X_n$  such that  $f(a) = a(x_1 \oplus x_2 \oplus \dots \oplus x_n)$  for all  $a \in I$ .

To show  $x_1 \oplus x_2 \oplus \dots \oplus x_n \in F = F_1 \oplus F_2$ .

Let  $U$  be neighborhood of zero of  $F$ , since the map  $f$  is continuous thus  $V = f^{-1}(U)$  neighborhood of zero of  $I$ , open of  $R$  that  $V$  is neighborhood of zero in  $R$ . By definition of  $f$  thus  $f(V) = V(x_1 \oplus x_2 \oplus \dots \oplus x_n)$  but  $f(V) = U$ , thus  $V(x_1 \oplus x_2 \oplus \dots \oplus x_n) \subset U$  that means  $x_1 \oplus x_2 \oplus \dots \oplus x_n \in F_1 \oplus F_2 \oplus \dots \oplus F_n = F$ .

**Proposition (3.12):**

If  $\oplus\{E_\alpha\}_{\alpha \in \Delta}$  be left topological module of topological ring  $R$  then  $\oplus\{E_\alpha\}_{\alpha \in \Delta}$  have injective envelope.

**Proof:**

On the same way of propositions (3.10) and (3.11).

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