CONTINUOUS DOUBLE-HYBRID POINT METHOD FOR THE SOLUTION OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

[1]Olanegan, O. O., [2]Awoyemi, D. O., [2]Ogunware, B. G. and [3]Obarhua, F. O.

[1, 2]Industrial Mathematics Department, Landmark University, Omu-Aran, Kwara State, Nigeria.

[2, 3]Mathematical Sciences Department, Federal University of Technology, Akure, Ondo State, Nigeria.

Corresponding Author: [1]Olanegan O. O.

E-mail: ola3yemi@yahoo.com, ola3posi@gmail.com

ABSTRACT

This research considers a continuous two-point hybrid for the general solution of second order ordinary differential equations with initial value problem (IVPs). The approximate solution is generated through power by the interpolation and collocation of the differential system. Taylor's series approximation was used to analyse and implement $y_{n+i}\ i = 1 \ldots n-1$ at $x_{n+i}, i = 0(1)2.$ The method is found to be consistent and zero-stable. The numerical results show better efficiency and accuracy compared to existing method of other authors.

Key words: *initial value problems, Power Series, Interpolation, collocation, Taylor's series, linear multistep method.*

1. INTRODUCTION

In this paper an initial value problems (IVPs) of general second order ordinary differential equations (ODEs) of the form:

$$y'' = f(x, y, y'), y(a) = y_0, y'(a) = \beta \qquad (1)$$

f is a real valued function which is continuous within the interval. f satisfied the Lipstich condition that guaranted the existence and the uniqueness of the solution equation (1). Many problems in (1) may not be easily solved analytically. Hence, numerical schemes are often developed to solve (1). We often reduced equation (1) to first order ordinary differential equations which are used to solve them. The reduction appraoch has been discussed by several authors. To avoid the rigour of reducing (1) to first order ordinary differential equations, authors proposed linear multistep method to solve equaion (1) directly. Among such authors are Awoyemi (1999 and 2001), Awoyemi and Kayode (2005), Adesanya *e tal*, (2009), Badmus and Yahaya (2009). According to Awoyemi (1999), continuous linear multistep method has greater advantages over the discrete method in that they give better error estimation, provide a simplified coefficient for further analytical work at different points, and guarantee easy appropriation of solution at all interior points of the integration interval. In Lambert (1973), he also discussed an optimal two step method called the numerov's method where he also solved it directly. Kayode (2011), proposed a three-step one point hybrid method based on collocation at selected grid and off-grid points. Among the authors that proposed continuous linear multistep methods are Awoyemi (2001), Onumanyi *e tal.* (1994), Adesanya *e tal* (2009), to mention a few. These authors individually implemented their methods with predictor-correct and block method, and adopted Taylor series expansion to supply starting value. According to Adesanya (2012), the setback of the predictor corrector method is that it is very costly as subroutines are very complicated to write because of the special techniques required to supply starting values and for varying the step size which leads to longer computer time and more human effort. The predictors they developed have reducing order predictors. Hence, it affects the accuracy of the method. In developing numerical methods, these factors must be taken into consideration.

a. The accuracy of the method
b. The cost of implementation of the method
c. Time taken to develop the method and
d. Flexibility of the method.

Also, various author developed the hybrid method. This method, while retaining certain charateristics of the continuous linear multistep method, share with Runge-Kutta methods the property of utilizing data at other points, other than the step point $x_{n+j}, j = 0, 1 \cdots n-1$. This method is useful in reducing the step number of a method and still remain zero stable. Since the predictor corrector method has not met the requirements above, hence there is a need to develop other method to cater for the draw-backs. Therefore, scholars developed block method to cater for the setback of predictor-corrector method. Among such authors are Awoyemi (2001), Jator and Li (2009), Majid, Suleiman and Omar (2006), they all proposed block methods. But, in Adesanya (2102), he stated that block method has a setback of not being able to exhaust all the possible interpolation points because the interpolation points cannot exceed the order of the differential equation. With all this drawback, Bolarinwa *e tal.* (2013), proposed Taylor series approximation method to improve on the setback usually faced with Predictor-Corrector and Block methods. In this research, we development a two point hybrid linear multistep method based on collocation at selected grid and off-grid point is developed for the direct solution of second order initial value problems of ordinary differential equation using hybrid method $x_{n+j,} j = 0, 1 \cdots n-1$ with Taylor's series being used to analyse and implement the method to improve on Bolarinwa *e tal* (2013).

2. The Method

A hybrid two-step implicit method is proposed as:

$$y(x) = \sum_{j=0}^{(r+i)-1} a_j x^j \qquad (2)$$

Which is a power series with a single variable x and $(r+i)$ is the sum of the collocation and interpolation points.

where a_j – parameters to be determined

r – number of collocation

i – number of interpolation

The first and second derivatives are

$$y'(x) = \sum_{j=0}^{(r+i)-1} j a_j x^{j-1} \tag{3}$$

$$y'' = j(j-1)a_j x^{j-2} \tag{4}$$

Combining (3) and (4) generates the differential system

$$\sum_{j=0}^{(r+i)-1} j(j-1)a_j x^{j-2} = f(x, y, y') \tag{5}$$

Collocating (5) at $x = x_{n+i}, i = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2$ and interpolating (2) at $x = x_{n+i}, i = 0, \tfrac{1}{2}$ and evaluating at the end point i.e $x = x_{n+i}, i = 1$.

The scheme result into a system of equation

$$\sum_{j=0}^{(r+i)-1} j(j-1)a_j x^{j-2} = f_{n+i} 0 \le i \le 1 \tag{6}$$

and

$$\sum_{j=0}^{(r+i)-1} a_j x^j = y_{n+i} 0 \le i \le 1 \tag{7}$$

where

$$x_{n+i} = x_n + ih$$

As stated above, the collocation and interpolation process results into the equation stated below:

$$\begin{aligned}
f_n &= 2a_2 + 6a_3 x_n + 12a_4 x_n^2 + 20a_5 x_n^3 \\
f_{n+\frac{1}{3}} &= 2a_2 + 6a_3 x_{n+1/3} + 12a_4 x_{n+1/3}^2 + 20a_5 x_{n+1/3}^3 \\
f_{n+2/3} &= 2a_2 + 6a_3 x_{n+2/3} + 12a_4 x_{n+2/3}^2 + 20a_5 x_{n+2/3}^3 \\
f_{n+1} &= 2a_2 + 6a_3 x_{n+1} + 12a_4 x_{n+1}^2 + 20a_5 x_{n+1}^3 \\
y_n &= a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5 \\
y_{n+1/3} &= a_0 + a_1 x_{n+1/3} + a_2 x_{n+1/3}^2 + a_3 x_{n+1/3}^3 + a_4 x_{n+1/3}^4 + a_5 x_{n+1/3}^5
\end{aligned} \tag{8}$$

Applying gaussian elimination method to the systems of equation (8), yields the value of $a's$ as follows:

$$a_5 = -\frac{9h^3}{40} f_n + \frac{27h^3}{40} f_{n+1/3} - \frac{27h^3}{40} f_{n+2/3} + \frac{9h^3}{40} f_{n+1}$$

$$a_4=\frac{3h^3}{8}\left(2h+3x_n\right)f_n-\frac{3h^3}{8}\left(5h+9x_n\right)f_{n+2/3}-\frac{3h^3}{8}\left(h+3x_n\right)f_{n+1}$$

$$a_3=\frac{h^3}{12}\left(11h^2+36x_nh+27x_n^2\right)f_n+\frac{h^3}{4}\left(2h^2+10x_nh+9x_n^2\right)f_{n+1/3}$$
$$-\frac{3h^3}{4}\left(h^2+8x_nh+9x_n^2\right)f_{n+2/3}+\frac{h^3}{12}\left(2h^2+18x_nh+27x_n^2\right)f_{n+1}$$

$$a_2=\frac{h^3}{4}\left(2h^3+11x_nh^2+18x_n^2h+9x_n^3\right)f_n-\frac{4hx_n^3}{9}\left(2h^2+5x_nh+3x_n^2\right)f_{n+1/3}$$
$$+\frac{4x_nh^3}{9}\left(h^2+4x_nh+3x_n^2\right)f_{n+2/3}-\frac{h^3}{4x_n}\left(2h^2+9x_nh+9x_n^2\right)f_{n+1}$$

$$a_1=\frac{h^3}{1080}\begin{pmatrix}97h^4+1080x_nh^3+2970x_n^2h^2+\\3240x_n^3h+1215x_n^4\end{pmatrix}f_n+\frac{h^3}{360}\begin{pmatrix}-38h^4+1620x_n^2h^2+\\2700x_n^3h+1215x_n^4\end{pmatrix}f_{n+1/3}$$
$$-\frac{h^3}{360}\begin{pmatrix}-13h^4+810x_n^2h^2\\+2160x_n^3h+1215x_n^4\end{pmatrix}f_{n+2/3}+\frac{h^3}{1080}\begin{pmatrix}-8h^4+540x_n^2h^2+\\1620x_n^3h+1215x_n^4\end{pmatrix}f_{n+1}$$
$$-\frac{3}{h}y_n+\frac{3}{h}y_{n+1/3}$$

$$a_0=y_n-a_1x_n-a_2x_n^2-a_3x_n^3-a_4x_n^4-a_5x_n^5$$

Substituting the values of a_i $i=0,1\cdots6$ into (1) results into equation of the form

$$y(x)=y_n+\sum_{i=1}^{6}a_i\left(x^i-x_n^i\right)$$

Now using the transformation:

$$t=\frac{x-x_n}{h}$$

$$\frac{dt}{dx}=\frac{1}{h}$$

becomes:

$$y_{n+1}+\left(3t\right)y_{n+1/3}+\left(1-3t\right)y_n=\frac{h^2}{1080}\begin{bmatrix}\left(-97t+540t^2-990t^3+810t^4-243t^5\right)f_n\\+\left(-38t+540t^3-675t^4+243t^5\right)f_{n+1/3}\\+\left(13t-270t^3+540t^4-243t^5\right)f_{n+2/3}\\+\left(-8t-180t^2-405t^3-405t^4+243t^5\right)f_{n+1}\end{bmatrix}$$

The coefficients are put as follows:

$$
\begin{aligned}
&\alpha_0 = 1-3t \\
&\alpha_{1/3} = 3t \\
&\beta_0 = \frac{h^2}{1080}\left[-97t+540t^2-990t^3+810t^4-243t^5\right] \\
&\beta_{1/3} = \frac{h^2}{360}\left[-38t+540t^3-675t^4+243t^5\right] \\
&\beta_{2/3} = \frac{h^2}{360}\left[13t-270t^3+540t^4-243t^5\right] \\
&\beta_1 = \frac{h^2}{1080}\left[-8t+180t^2-405t^4+243t^5\right]
\end{aligned} \tag{9}
$$

The first derivatives of (9) are as follows:

$$
\begin{aligned}
&\alpha_0' = -\frac{2}{h} \\
&\alpha_{1/3}' = \frac{3}{h} \\
&\beta_0' = \frac{h}{1080}\left[-97+1080t-2970t^2+3240t^3-1215t^4\right] \\
&\beta_{1/3}' = \frac{h}{360}\left[-38+1620t^2-2700t^3+1215t^4\right] \\
&\beta_{2/3}' = \frac{h}{360}\left[13-810t^2+2160t^3-1215t^4\right] \\
&\beta_1' = \frac{h}{1080}\left[-8+360t-1215t^2-1620t^3+1215t^4\right]
\end{aligned} \tag{10}
$$

Evaluating (9) and (9.1) at $t=1$ which implies that $x = x_{n+1}$ gives our scheme as:

$$
y_{n+1} - 3y_{n+1/3} + 2y_n = \frac{h^2}{108}\left[2f_n + 21f_{n+1/3} + 12f_{n+2/3} + f_{n+1}\right] \tag{11}
$$

With the order $c_5 = 4,$ Error constant $c_6 = -\dfrac{1}{58320}$

The first derivative is given by:

$$
y_{n+1}' = \frac{1}{h}\left(-2y_n + 2y_{n+1/3}\right) + \frac{h}{1080}\left(1268f_{n+1} + 444f_{n+2/3} + 291f_{n+1/3} + 38f_n\right) \tag{12}
$$

2.1 Taylor's Series Expansion of y Variables

Since we adopted Taylor's series for our approximation and implementation, there is a need for Taylor series expansion in this paper to be able to approximate y variables to generate values for the approximate solution of the scheme, the second derivative is expanded term by term up to the order of the scheme developed by Taylor's series expansion.

$$y_{n+i} = y\left(x_n + ih\right) = y_n + ihy'\left(x_n\right) + \frac{(ih)^2}{2!} f_n + \frac{(ih)^3}{3!} + \frac{(ih)^4}{4!} + \cdots \tag{13}$$

$$y'_{n+i} = y'\left(x_n\right) + ihf_n + \frac{(ih)^2}{2!} f'_n + \cdots \tag{14}$$

Then, we have

$$f_{n+i} = y''\left(x_n + ih\right)$$

$$y''\left(x_n + ih\right) = f_n + ihf'_n + \frac{(ih)^2}{2!} f''_n + \frac{(ih)^3}{3!} f'''_n + \frac{(ih)^4}{4!} f_n^{iv} + \cdots \tag{15}$$

where

$$f_n = f\left(x_n, y_n, y'_n\right)$$

$$f_n^{(i)} = f^{(i)}\left(x_n, y_n, y'_n\right), i = 1, 2, 3 \cdots$$

We find f', f'' and f''' f^{iv} by the use of partial derivatives as demonstrated below:

$$f' = \frac{df}{dx} = \frac{\partial f}{\partial x} + y'_n \frac{\partial f}{\partial y} + f \frac{\partial f}{\partial y'} \tag{16}$$

$$f'' = \frac{d^2 f}{dx^2} = 2\left(Ay' + Bf\right) + Cfy' + D + E \tag{17}$$

$$f''' = \frac{d^3 f}{dx^3} = 2G + 3\left(Hy' + If\right) + Jfy' + K + L + M \tag{18}$$

$$f^{iv} = \frac{d^4 f}{dx^4} = N + 4fO + Pf' + Q\left(y'\right)^2 + R + S + T + U + V + W \tag{19}$$

where

$$A = \frac{\partial^2 f}{\partial x \partial y} + f \frac{\partial^2 f}{\partial y \partial y'}$$

$$B = \frac{\partial^2 f}{\partial x \partial y'}$$

$$C = \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + f \frac{\partial f}{\partial y'}$$

$$D = \frac{\partial^2 f}{\partial x^2} + \left(y'\right)^2 \frac{\partial^2 f}{\partial y^2} + f^2 \frac{\partial^2 f}{\left(\partial y'\right)^2}$$

$$E = f \frac{\partial y}{\partial y}$$

$$G = y'f' \frac{\partial^2 f}{\partial y \partial y'} + f' \frac{\partial^2 f}{\left(\partial y'\right)^2} + y'ffy' \frac{\partial^2 f}{\partial y \partial y'} + f' \frac{\partial^2 f}{\partial x \partial y}$$

$$H = \frac{\partial^3 f}{\partial x \partial y} + y' \frac{\partial^3 f}{\partial x \partial y^2} + f \frac{\partial^2 f}{\partial y^2} + y'f \frac{\partial^3 f}{\partial y^2 \partial y'} + f^2 \frac{\partial^3 f}{\partial y \left(\partial y'\right)^2} + 2 \frac{\partial^3 f}{\partial x \partial y \partial y'}$$

$$I = \frac{\partial^3 f}{\partial x^2 \partial y'} + \frac{\partial^2 f}{\partial x \partial y} + f \frac{\partial^3 f}{\partial y \left(\partial y'\right)^2} + f \frac{\partial^2 f}{\partial y \partial y'}$$

$$J = f \frac{\partial f}{\partial y} + \partial y' \frac{\partial^2 f}{\partial x \partial y} + f \frac{\partial^3 f}{\partial y \left(\partial y'\right)^2} + f \frac{\partial^2 f}{\partial y \partial y'}$$

$$K = \left(\frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + f \frac{\partial f}{\partial y'} \right) \left[\frac{\partial^2 f}{\partial x \partial y'} + y' \frac{\partial^2 f}{\partial y \partial y'} + f \frac{\partial^2 f}{\left(\partial y'\right)} \right]$$

$$L = \frac{\partial^3 f}{\partial x^3} + f^3 \frac{\partial^3 f}{\left(\partial y'\right)^3} + \left(y'\right)^3 \frac{\partial^3 f}{\partial y^3}$$

$$M = f' \frac{\partial f}{\partial y}$$

$$N = \left(y'\right)^3 \frac{\partial^4 f}{\partial y^4} + f^5 \frac{\partial^4 f}{\left(\partial y'\right)^4}$$

$$O = 4 f \left[\frac{\partial^4 f}{\partial x^2 \partial y} + \frac{\partial^4 f}{\partial x^3 \partial y'} \right]$$

$$P = f' \left[\begin{array}{l} 2 \frac{\partial^2 f}{\left(\partial y'\right)^2} + \left(f'\right) \frac{\partial^2 f}{\left(\partial y'\right)^2} + 2 \left(f'\right) \frac{\partial^2 f}{\partial x \partial y} + 9 y' \\ \frac{\partial^3 f}{\partial x \partial y \partial y'} + 6 \frac{\partial^3 f}{\partial x^2 \partial y'} + \\ 6 \left(y'\right)^2 \frac{\partial^3 f}{\partial y^2 \partial y'} + 2 y' \frac{\partial^2 f}{\partial y} + \\ 2 \frac{\partial^2 f}{\partial x \partial y} + 4 f \frac{\partial^2 f}{\partial y \partial y'} + \\ 4 f y' \frac{\partial^3 f}{\partial x \left(\partial y'\right)^2} + 2 f \frac{\partial^3 f}{\partial x^2 \partial y} \end{array} \right]$$

$$Q = \left(y'\right)^2 \left[5 \frac{\partial^3 f}{\partial y^3} + 4 \left(y'\right) \frac{\partial^4 f}{\partial x \partial y^3} + 4 f \left(y'\right) \frac{\partial^4 f}{\partial y^3 \partial y'} + 6 \frac{\partial^4 f}{\partial x^2 \partial y^2} \right]$$

$$R = 2y'\begin{bmatrix} 6f\dfrac{\partial^4 f}{\partial x^2\partial y\partial y'} + 4\dfrac{\partial^3 f}{\partial x\partial y^2} + 4f\dfrac{\partial^3 f}{\partial y^2\partial y'} + \\ 6f\left(y'\right)\dfrac{\partial^4 f}{\partial x\partial y^2\partial y} + 3f\left(y'\right)\dfrac{\partial^4 f}{\partial y^2\left(\partial y'\right)^2} + \\ 4f\dfrac{\partial^3 f}{\partial y\left(\partial y'\right)^2} + 6f^2\dfrac{\partial^4 f}{\partial x\partial y\left(\partial y'\right)^2} + \dfrac{\partial^3}{\partial x\partial y^2} + \\ ff'\dfrac{\partial^3 f}{\partial x\partial y} + f\left(y'\right)\dfrac{\partial^3 f}{\partial y^3} + 4f^2\dfrac{\partial^3 f}{\partial y^2\partial y'} \end{bmatrix}$$

$$S = 2f^2\begin{bmatrix} 5\dfrac{\partial^3 f}{\partial x\partial y\partial y'} + 2f\dfrac{\partial^4 f}{\partial x\left(\partial y'\right)^3} + 2fy'\dfrac{\partial^4 f}{\partial y\left(\partial y'\right)^3} + \\ 3\dfrac{\partial^4 f}{\partial x^2\left(\partial y'\right)^2} + \dfrac{\partial^2 f}{\partial y^2} + f'\dfrac{\partial^3 f}{\left(\partial y'\right)^3} + \\ f'\dfrac{\partial^3 f}{\partial x\partial y\partial y'} + 3f\dfrac{\partial^4 f}{\partial y(\partial y')^2} \end{bmatrix}$$

$$T = 2f\left[3\frac{\partial^2 f}{\partial y\partial y'} + \frac{\partial^2 f}{\partial y^2} + 4\frac{\partial^3 f}{\partial x\left(\partial y'\right)^2}\right]$$

$$U = f''\left[\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial x\partial y'} + 4f'\frac{\partial^2 f}{\partial y\partial y'} + f\frac{\partial^2 f}{\left(\partial y'\right)^2} + 2f\frac{\partial^2 f}{\partial x^2\partial y}\right]$$

$$V = 4y'\frac{\partial^4 f}{\partial x^3\partial y}$$

$$W = f'''\frac{\partial f}{\partial y'}$$

3. Basic Properties and Analysis of the Method

3.1 Order and Error Constant of the Method

This paper adopt the method proposed by Lambert (1973), with the linear operator:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h^2\sum_{j=0}^{k}\beta_j f_{n+j}$$

We associate the linear operator L with the scheme and define as

$$L\left\{y\left(x\right),h\right\} = \sum_{j=0}^{k}\left[\alpha_j y\left(x+jh\right) - h^2\beta_j y''\left(x+jh\right)\right]$$

where α_0 and β_0 are both non-zero and $y(x)$ is an arbitrary function which is continuous and differentiable on the interval [a, b]. If we assume that $y(x)$ has as many higher derivatives as we require, then on Taylor's series expansion about the point x, we obtain

$$L\left[y(x,h)\right]=c_0 y(x)+c_1 hy'(x)+\cdots+cqh^q y^q(x)+\cdots$$

Accordingly we say that the method has order P if,

$$c_0=c_1=\cdots=c_p=c_{p+1}=0, c_{p+2}\neq 0$$

Then, c_{p+2} is the error constant and $C_{p+2}h^{p+2}y^{p+2}(x_n)$ is the principal local truncation error at the point x_n .

In this paper, since $c_0=c_1=c_2=\cdots=c_6$ and $c_7=c_{p+2}\neq 0$ which implies that the scheme is of order 4 and the error constant $c_{p+2}=-\frac{1}{58320}$ or -1.715×10^{-6}

3.2 Consistency of the Method

For a method to be consistent, the following criteria must be satisfied.

$$y_{n+1}-3y_{n+1/3}+2y_n=\frac{h^2}{108}\left[2f_n+21f_{n+1/3}+12f_{n+2/3}+f_{n+1}\right]$$

Condition 1: $p\geq 1$

The order is 4. Therefore, condition (1) is satisfied.

Condition 2: $\sum_{j=0}^{k}\alpha_j=0$ where $j=(0\cdots 1)$

$$\alpha_0+\alpha_{1/3}+\alpha_{2/3}+\alpha_1=0$$

$$2-3+0+1=0$$

This also satisfy condition (2)

Condition 3: $\rho'(r)=0$ when $r=1$

$$\rho=r-3r^{1/3}+2$$

$$\rho'=1-r^{-2/3}=0 \text{ when } r=1$$

This method also satisfy condition 3.

Condition 4: $\rho''(r)=2!\sigma(r)$ when $r=1$

$$\rho''=\frac{2}{3}r^{-1/3} \quad r=1, \quad \rho''=\frac{2}{3}$$

$$\sigma(r)=\frac{1}{108}\left(r+12r^{2/3}+21r^{1/3}+2\right)$$

$$\sigma(1)=\frac{1}{3}$$

and $2!\sigma(1) = 2\left(\frac{1}{3}\right) = \frac{2}{3}$

Since $\rho''(r) = 2!\sigma(r)$ at $r = 1$, then the condition (4) is satisfied.
Thus, the method is said to be consistent.

3.3 Zero Stability

Definition (Lambert [9]): A linear multistep method is said to be zero-stable, if no root of the first characteristics polynomials $\rho(r)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than two.

Our method is zero stable since no root of the first characteristics polynomial $\rho(r)$ has modulus greater than one that is $|r| \leq 1$. This implies that the method is zero stable if,

$\sum_{j=0}^{k} \alpha_j = 0$, where α_j are the coefficients of $\sum_{j=0}^{k} \alpha_j y_{n+j}$

$$\sum_{j=0}^{k} \alpha_j = \alpha_1 - 3\alpha_{2/3} + 2\alpha_0 = 1 - 3 + 2 = 0$$

3.4 Convergence

Thus, since the method developed satisfies the necessary and sufficient condition for a linear multistep method to be convergent (consistency and zero stability), then our method is said to be convergent.

4. Method Implementation with Numerical Examples

4.1 Test Problem

Problem 1

$$y'' = x(y')^2 \quad y(0) = 1,\ y'(0) = \frac{1}{2},\ h = \frac{1}{320}$$

Theoretical Solution:

$$y(x) = 1 + \frac{1}{2}\log\left(\frac{2+x}{2-x}\right)$$

Problem 2

$$y'' = y', \qquad y(0), y'(0) = -1\ ,\ h = 0.01$$

Exact Solution

$y(x) = 1 - e^x$

4.2 Results and Interpretation

Table 1: Numerical Result of Problem 1 compared with Bolarinwa (2013) and Ehigie (2010)

X	**Computed Solution**	**Error in Bolarinwa *e tal* (2013)** $(p=5, k=1)$	**Error in Ehigie *e tal*** $(p=6, k=2)$	**Error in our new method** $(p=4, k=1)$
0.2	1.100335347731237000	1.5863E-12	8.46E-06	1.616485E-13
0.4	1.202732554054486400	6.6722E-12	1.66E-05	4.047873E-13
0.6	1.309519604203716900	1.7757E-11	2.41E-05	6.050715E-13
0.8	1.423648930193764000	4.8314E-11	3.07E-05	1.605382E-13
1.0	1.549306144332172500	9.1080E-11	3.60E-05	1.886047E-12

Table 2: Numerical Result of Problem 2 compared with Bolarinwa (2013)

X	**Computed Solution**	**Error in Bolarinwa e tal (2013)** $(p=5, k=1)$	**Error in our new Method** $(p=4, k=1)$
0.2	-0.105170618075	9.0153E-8	1.6425E-12
0.4	-0.349657807571	3.9781E-7	8.6628E-12
0.6	-0.648721070685	1.0149E-6	2.0317E-11
0.8	-1.003752507436	2.0633E-6	4.5463E-11
1.0	-1.459603111094	3.7037E-6	8.2113E-11

Table 3: Numerical Result of Problem 2 compared with Adesanya (2011)

X	**Computed Solution**	**Error in Adesanya (2011)** $(p=5,k=2)$	**Error in our new method** $(p=4,k=1)$
0.1	-0.10517091	6.4420E-11	2.2360E-013
0.2	-0.22140275	5.4567E-10	1.6425E-012
0.3	-0.34985880	1.9216E-09	3.4625E-012
0.4	-0.49182469	4.7970E-09	8.6628E-012
0.5	-0.64872127	9.9980E-09	1.1338E-011
0.6	-0.82211880	1.8714E-08	2.0317E-011
0.7	-1.01375270	3.2728E-08	3.2476E-011
0.8	-1.22554092	5.4792E-08	4.5463E-011
0.9	-1.45960311	8.9294E-08	6.1781E-011
1.0	-1.71828182	1.4347E-07	8.2113E-011

The numerical results to the problem 1 is as shown in the table 1, compared with the results obtained by Bolarinwa e *tal* (2013) and Ehigie *e tal* (2010) of order 5 and 6 respectively. Bolarinwa e *tal* (2013) was implemented with Taylor's approach while Ehigie *e tal* (2010) was implemented in block mode.
The Numerical result to the problem 2 is as shown in the table 2 and 3, compared with the results obtained by Bolarinwa e tal (2013) and Adesanya (2010) of order 5 respectively. Adesanya (2011) was implemented in block mode.

5. Conclusion

From the results shown in the table above, in Table 1, our method perform better than the method of Bolarinwa *e tal* with ($p=5,k=1$) implemented with Taylor's series approach and Ehigie e *tal* despite ($p=6,k=2$) which was implemented in block method.
Likewise, table 2 and table 3 for problem 2 there is better accuracy in our method compared to other authors, with ($p=5,k=1$) implemented in Taylor series and Adesanya (2011) $(p=5,k=2)$ implemented in block as demonstrated by both authors.

From the above results, the method derived in this research, has been able to solve general second order differential equations.

Thus, the method derived in this research has been able to solve general second order ordinary differential equations with initial value problems directly. It is worth noting that the new method in this paper is efficient and compared favourably with existing methods.

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