Solution of System of Differential Equations with Initial Conditions Using Repeated Interpolation Technique

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Abstract

The aim of this paper is to present a method for solving a system of first order ordinary differential equation with initial conditions by repeated interpolation technique with constructing polynomial solutions. The original problem is concerned using two-point osculatory interpolation with the fit equal numbers of derivatives at the end points of an interval [0 , 1] .

The existence, uniqueness of solutions to the system of ordinary initial value problems is discussed. Also, introduce some general observations about control of a residual and control of the local error and develop inexpensive estimates of both the local error and the size of the residual. Then develop an effective program to solve the system of ordinary initial differential equations.

1. Introduction

Many problems in engineering and science can be formulated in terms of differential equations. A differential equation is an equation involving a relation between an unknown function and one or more of its derivatives. An ordinary differential equation (ODE) has only one independent variable , and all derivatives in it are taken with respect to that variable . Most often, this variable is time t. Gerald and Wheatley flip around a lot , using both t and x as the independent variable ; pay careful attention to what the derivative is taken with respect to so you don't get confused.

The problems of solving an ODE are classified into initial value problems (IVP) and boundary value problems (BVP), depending on how the conditions at the endpoints of the

domain are specified. All the conditions of an initial value problem are specified at the initial point. On the other hand, the problem becomes a boundary value problem if the conditions are needed for both initial and final points. [1]

A general mth-order initial value problem :

$y^{(m)} = f(x , y, y', y'', \dots , y^{(m-1)}) , \quad a \le x \le b$ (1)

With initial conditions :

$$y(a) = \alpha_1, y'(a) = \alpha_2, \dots, y^{(m-1)}(a) = \alpha_m \quad (2)$$

The system of m first-order differential equations (meaning that only the first derivative of y appears in the equation and no higher derivatives) have the form :

$$y'_1 = f_1 (x , y_1 , \dots, y_m)$$
$$y'_2 = f_2 (x , y_1 , \dots, y_m)$$
$$\vdots$$
$$y'_m = f_m(x, y_1, y_2, \dots, y_m) \quad (3a)$$

with initial conditions :

$$y_1(a) = \alpha_1 , y_2(a) = \alpha_2 , \dots , y_m(a) = \alpha_m \quad (3b)$$

It is easy to see that (3) can represent either an mth-order differential equation, a system of equations of mixed order but with total order of m , or system of m first -order equations.

The IVP is said to be homogeneous if both the differential equation and the initial conditions are homogeneous. Otherwise the problem is non homogeneous .Initial value problem is solved by first obtaining the general solution to the differential equation, using any of the appropriate methods presented heretofore, and then applying the initial conditions to evaluate the arbitrary constants. [2]

Since there are relatively few differential equations arising from practical problems for which analytical solutions are unknown, one must resort to numerical methods. In this paper we study the solutions of IVP , where the problems define on the interval [0 , 1] using semi – analytic methods that give solution with high accuracy and easy implemented from other numerical methods .

2. Existence and Uniqueness of System of IVP

To discuss existence and uniqueness of solutions to systems of equations, we need to extend the definition of the Lipschitz condition to functions of several variables .

Definition 1

The function $f(x, y_1, \dots y_m)$, defined on the set :

$$D = \{(x, u_1, ..., u_m): a \le x \le b, -\infty < u_i < \infty, \ for each i = 1,2,...m\}$$

Is said to satisfy a Lipschitz condition on D in the variables $u_1, u_2, ..., u_m$ if a constant $L > 0$ exists with :

$$|f(x, u_1, ..., u_m) - f(x, y_1, ..., y_m)| \le L \sum_{j=1}^{m} |u_j - y_j| \qquad (4)$$

For all $(x, u_1, ..., u_m)$ and $(x, y_1, ..., y_m)$ in D .

By using the mean value theorem, it can be shown if f and its first partial derivatives are continuous on D and if :

$$\left| \frac{\partial f(x, u_1, ..., u_m)}{\partial u_i} \right| \le L,$$

For each $i = 1,2,..., m$ and all $(x, u_1, ..., u_m)$ in D, then f satisfies a Lipschitz condition on D with Lipschitz constant L (see [3]). A basic existence and uniqueness theorem follows. Its proof can be found in [3] .

Theorem 2

Suppose

$$D = \{(x, u_1, u_2, ..., u_m) | a \le x \le b, -\infty < u_i < \infty, \ for each i = 1,2,..., m\}$$

and let $f_i(x, u_1, ..., u_m)$, for each $i = 1,2,...m$, be continuous on D and satisfy a Lipschitz condition. The system of first-order differential equations (3a), subject to the initial conditions (3b), has a unique solution $u_1(x), ..., u_m(x)$, for $a \le x \le b$.

Methods to solve systems of first-order differential equations are generalizations of the methods for a single first-order differential equation.

3. Approximation Theory

The primary aim of a general approximation is to represent non-arithmetic quantities by arithmetic quantities so that the accuracy can be ascertained to a desired degree. Secondly, we are also concerned with the amount of computation required to achieve this accuracy. A complicated function f(x) usually is approximated by an easier function of the form $\varphi(x; a_0, \ldots, a_n)$ where $a_0, \ldots, a_n$ are parameters to be determined so as to characterize the best approximation of f .

In this paper, we shall consider only the interpolatory approximation . From Weierstrass Approximation Theorem ,it follows that one can always find a polynomial that is

arbitrarily close to a given function on some finite interval. This means that the approximation error is bounded and can be reduced by the choice of the adequate polynomial. Unfortunately Weierstrass Approximation Theorem is not a constructive one, i.e. it does not present a way how to obtain such a polynomial. i.e. the interpolation problem can also be formulated in another way, viz. as the answer to the following question: How to find a .good. representative of a function that is not known explicitly, but only at some points of the domain of interest .In this paper we use Osculatory Interpolation since has high order with the same given points in the domain.

3.1. Osculatory Interpolation [3]

Given the data $\{x_i\}$, i = 0,1, . . .,n and values $f_i^{(0)}, \ldots, f_i^{(mi)}$,where m_i are nonnegative integers and $f_i = f(x_i)$ .We want to construct a polynomial P(x) such that :

$$P^{(j)}(x_i) = f_i^{(j)} \qquad (5)$$

For each i = 0, 1, . . . , n and j = 0, . . . , m_i .

i.e. the osculating polynomial approximating a function $f \in C^m [a , b]$, where $m = \max \{m_0 , m_1 , \ldots , m_n \}$ and $x_i \in [a , b]$, for each i = 0, 1, . . . , n . Such a polynomial is said to be an osculatory interpolating polynomial of a function f .

There exist various form for osculatory interpolation ,but all of these differed only in formula ,the following theorem illustrate this :

Theorem 3 [4],[6],[5]

Given the nodes $\{x_i\}$, i = 0, . . . , n and values $\{ f_i^{(j)} \}$, j = 0, . . . , m_i ,

there exists a unique polynomial satisfying (5).

For this method we have the following error bound :

Theorem 4 [7]

Suppose $f \in C^m [a, b]$. Let the interpolation nodes satisfy : $a \le x_0 < x_2 < \ldots < x_n \le b$, the polynomial P interpolates f at $\{x_i\}_{i=0}^n$ and $w(x) = \prod_{i=0}^{n} (x - x_i)$. Then there exists $\zeta \in [a, b]$ such that the error function R(x) satisfies :

$$R(x) := f(x) - P(x) = (f^{((m+1)n+m+1)} (\zeta) / ((m+1)n + m+1)!) w(x)^{m+1} \quad \ldots\ldots (6)$$

In this paper we use two-point osculatory interpolation [8]. Essentially this is a generalization of interpolation using Taylor polynomials and for that reason osculatory

interpolation is sometimes referred to as two-point Taylor interpolation. The idea is to approximate a function y(x) by a polynomial P(x) in which values of y(x) and any number of its derivatives at given points are fitted by the corresponding function values and derivatives of P(x) .

In this paper we are particularly concerned with fitting function values and derivatives at the two end points of a finite interval, say [0 , 1] ,wherein a useful and succinct way of writing osculatory interpolant $P_{2n+1}(x)$ of degree 2n + 1 was given for example by Phillips [9] as :

$$P_{2n+1}(x) = \sum_{j=0}^{n} \{ y^{(j)}(0)\, q_j(x) + (-1)^j\, y^{(j)}(1)\, q_j(1-x) \} \qquad (7)$$

$$q_j(x) = (x^j / j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s = Q_j(x) / j! \qquad (8)$$

so that (7) with (8) satisfies :

$y^{(j)}(0) = P_{2n+1}^{(j)}(0)$, $y^{(j)}(1) = P_{2n+1}^{(j)}(1)$, j = 0, 1, 2,…, n .

implying that $P_{2n+1}(x)$ agrees with the appropriately truncated Taylor series for y(x) about x = 0 and x = 1. The error on [0, 1] is given by :

$R_{2n+1} = y(x)-P_{2n+1}(x) = \dfrac{(-1)^{n+1} x^{(n+1)} (1-x)^{n+1} y^{(2n+2)}(\varepsilon)}{(2n+2)!}$ where $\varepsilon \in (0, 1)$ and $\mathbf{y}^{(2n+2)}$ is assumed to be continuous.

The osculatory interpolant for $P_{2n+1}(x)$ may converge to y(x) in [0 , 1] irrespective of whether the intervals of convergence of the constituent series intersect or are disjoint .The important consideration here is whether $R_{2n+1} \rightarrow 0$ as $n \rightarrow \infty$ for all x in [0 ,1]. In the application to the IVP in this paper such convergence with n is always confirmed numerically .We observe that (7) fits an equal number of derivatives at each end point but it is possible and indeed sometimes desirable to use polynomials which fit different numbers of derivatives at the end points of an interval .

Finally we observe that (7) can be written directly in terms of the Taylor coefficients a_i and b_i about x = 0 and x = 1 respectively, as :

$$P_{2n+1}(x) = \sum_{j=0}^{n} \{ a_j Q_j(x) + (-1)^j b_j Q_j(1-x) \} \qquad (9)$$

4. Solution of System of Ordinary Differential Equation with Initial Conditions

We consider the system of two equations of first order ordinary differential equations with initial conditions of the form:

$y_1' = dy_1 / dx = f_1(x, y_1, y_2)$

$y_2' = dy_2 / dx = f_2(x, y_1, y_2)$, (10a)

For $0 \le x \le 1$, with the initial conditions :

$y_1(0) = a_0$, $y_2(0) = b_0$, (10b)

where f_i , i = 1, 2 are in general nonlinear functions of their arguments .

The simple idea behind the use of two-point polynomials is to replace y(x) in problem (10a) – (10b), or an alternative formulation of it, by a P_{2n+1} (defined in section 3) which enables any unknown derivatives of y(x) to be computed . The first step therefore is to construct the P_{2n+1} . To do this we need the Taylor coefficients of $y_1(x)$ and $y_2(x)$ respectively about x = 0 :

$$y_1 = a_0 + a_1 x + \sum_{i=2}^{\infty} a_i x^i \quad (11a)$$

$$y_2 = b_0 + b_1 x + \sum_{i=2}^{\infty} b_i x^i \quad (11b)$$

where $y_1(0) = a_0$, $y_1'(0) = a_1$, … , $y_1^{(i)}(0) / i! = a_i$, i = 2, 3, ……

and $y_2(0) = b_0$, $y_2'(0) = b_1$, … , $y_2^{(i)}(0) / i! = b_i$, i = 2, 3, ……

then insert the series forms (11a) and (11b) respectively into (10a) and equate coefficients of powers of x .

Also ,we need Taylor coefficients of $y_1(x)$ and $y_2(x)$ about x = 1, respectively

$$y_1 = c_0 + c_1(x-1) + \sum_{i=2}^{\infty} c_i (x-1)^i \quad (12a)$$

$$y_2 = d_0 + d_1(x-1) + \sum_{i=2}^{\infty} d_i (x-1)^i \quad (12b)$$

where $y_1(1) = c_0$, $y_1'(1) = c_1$, … , $y_1^{(i)}(1) / i! = c_i$, i = 2, 3, ……

and $y_2(1) = d_0$, $y_2'(1) = d_1$, … , $y_2^{(i)}(1) / i! = d_i$, i = 2, 3, ……

then insert the series forms (12a) and (12b) respectively into (10a) and equate coefficients of powers of (x – 1) .

The resulting system of equations can be solved using MATLAB version 7.9 to obtain a_i , b_i , c_i and d_i for all $i \geq 2$, we see that c_i's and d_i's coefficients depend on indicated unknowns c_0 and d_0 .

The algebraic manipulations needed for this process .We are now in a position to construct a $P_{2n+1}(x)$ and $\widetilde{P}_{2n+1}(x)$ from (11) and (12) of the form (7) by the following :

$$P_{2n+1}(x) = \sum_{i=0}^{n} \{ a_i Q_i(x) + (-1)^i c_i Q_i(1-x) \} \qquad (13a)$$

and

$$\widetilde{P}_{2n+1}(x) = \sum_{i=0}^{n} \{ b_i Q_i(x) + (-1)^i d_i Q_i(1-x) \} \qquad (13b)$$

Where $Q_i(x)$ defined in (8) ,

We see that (13) have only two unknowns c_0 and d_0 .

Now, integrate equation (10a) to obtain :

$$c_0 - a_0 = \int_0^1 f_1(x, y_1, y_2) \, dx \qquad (14a)$$

$$d_0 - b_0 = \int_0^1 f_2(x, y_1, y_2) \, dx \qquad (14b)$$

use P_{2n+1} and $\widetilde{P}_{2n+1}$ as a replacement of y_1 and y_2 respectively in (14) .

Since we have only the two unknowns c_0 and d_0 to compute for any n we only need to generate two equations from this procedure as two equations are already supplied by (14) and initial condition (10b). Then solve this system of algebraic equations using MATLAB version 7.9 to obtain c_0 and d_0 ,so insert it into (13) thus (13) represent the solution of (10) .

Extensive computations have shown that this generally provides a more accurate polynomial representation for a given n .

Use the same manner to solve in general the system of more than two equation .

Now consider the following example designed to illustrate the convergence, accuracy, implementation and utility of the method. In what follows the use of bold digits in the tables is intended to give a rough visual indication of the convergence.

Example

Linear initial value problems (IVPs) can be used to model several physical phenomena. For example, Kirchhoff's law states that the sum of all instantaneous voltage changes around a closed circuit is zero. This law implies that the current I(t) in a closed

circuit containing a resistance of R ohms, a capacitance of C farads, an inductance of L henries, and a voltage source of E(t) volts satisfies the equation :

$$LI'(t) + RI(t) + \frac{1}{C}\int I(t)\,dt = E(t).$$

The currents $I_1(t)$ and $I_2(t)$ in the left and right loops, respectively, of the circuit shown in figure 1 are the solutions to the system of equations :

$$2I_1(t) + 6[I_1(t) - I_2(t)] + 2I'_1(t) = 12$$
$$\frac{1}{0.5}\int I_2(t)\,dt + 4I_2(t) + 6[I_2(t) - I_1(t)] = 0$$

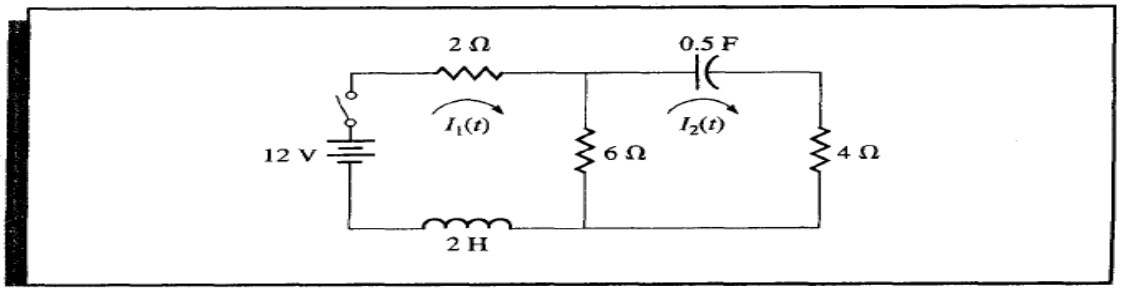

Figure 1

Suppose that the switch in the circuit is closed at time $t = 0$, the $I_1(0) = 0$ and $I_2(0) = 0$, then differentiating the second equation, and substituting the first equation (for $I'_1(t)$) into the resulting equation gives the system :

$$I'_1 = f_1(t, I_1, I_2) = -4I_1 + 3I_2 + 6, \quad I_1(0) = 0,$$
$$I'_2 = f_2(t, I_1, I_2) = 0.6I'_1 - 0.2I_2 = -2.4I_1 + 1.6I_2 + 3.6, \quad I_2(0) = 0$$

This problem is linear system IVP with exact solution is : [3]

$$I_1(t) = -3.375e^{-2t} + 1.875e^{-0.4t} + 1.5$$
$$I_2(t) = -2.25e^{-2t} + 2.25e^{-0.4t}$$

Now, we solve this system using semi-analytic method by the following :

From equations (7) and (8) we have :

$P_5 = 0.3647333436t^5 - 1.8223515054t^4 + 4.3583871465t^3 - 6.599999999977t^2 + 6t$

$P_7 = 0.033834086t^7 - 0.2368315955t^6 + 0.86262205699\,t^5 - 2.2395413784t^4 + 4.48t^3 - 6.5999999999976717t^2 + 6t$

$P_9 = 0.00185108897t^9 - 0.01665978077t^8 + 0.0814473547t^7 - 0.2980148375t^6 + 0.899469782\,t^5 - 2.24800000002t^4 + 4.48t^3 - 6.599999999977t^2 + 6t$

$\tilde{P}_5 = 0.2430854177t^5 - 1.2138488335t^4 + 2.8949294406\,t^3 - 4.3199999999t^2 + 3.6\,t$

$\tilde{P}_7 = 0.0225557904t^7 - 0.1578821252t^6 + 0.5749960924t^5 - 1.4919609323t^4 + 2.976t^3 - 4.3199999999488t^2 + 3.6t$

$\tilde{P}_9 = 0.0012340587t^9 - 0.0111065045t^8 + 0.05429791165t^7 - 0.1986708696t^6 +$

$$0.59956118804t^5 - 1.497600000002t^4 + 2.976t^3 - 4.31999999994t^2 + 3.6t$$

For more details ,table (1) gives the results of different nodes in the domain, for n = 2, 3, 4. Table (2) give a comparison between the different methods to illustrate the accuracy of suggested method . Also, figure (2) gives the accuracy of the suggested method .

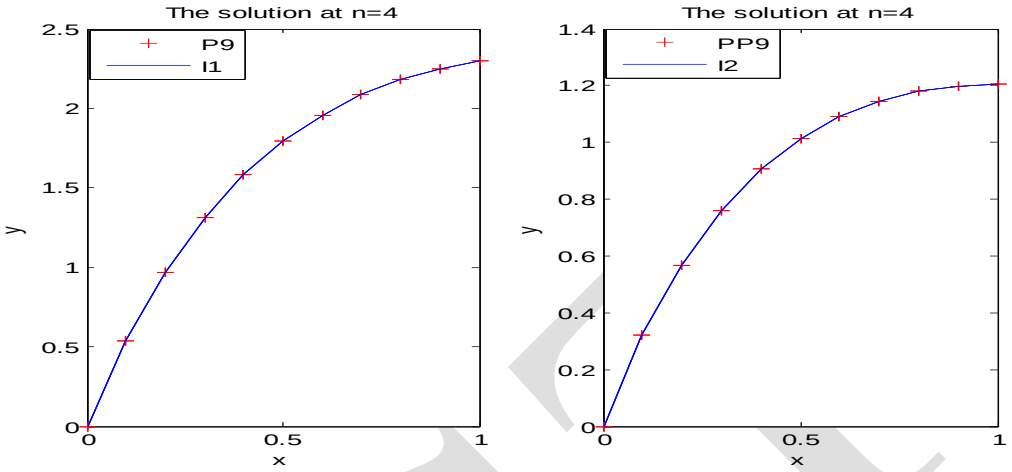

Figure 2: Comparison between the exact and semi-analytic solution P_9

Table 1 : The result of the method for n = 2, 3, 4 of example

		P_5	P_7	P_9	
c_o		2.30076898467955	2.30008316903662	2.30009360750152	
d_o		1.20416602485870	1.20370882540034	1.20371578436998	
t	I_1:exaxt	P_5	P_7	P_9	$\lvert I_1-P_9\rvert$
0	0	0	0	0	0
0.1	0.53826390677248	0.53817979932959	0.538264438634776	0.53826390466320	2.10921502485917e-09
	0.96851299410466	PP_5	PP_7	PP_9	3.58982520376472e-08
0.3	1.31073654702733	1.30980170778782	1.31075063571701	1.31073641467898	1.32348350279088e-07
0.4	1.58128435041602	1.58001944827763	1.58130636213144	1.58128411224080	2.38175224209769e-07
0.5	1.79352704806760	1.79229934121385	1.79354943825329	1.79352679445575	2.53611845080215e-07
0.6	1.95839677429611	1.95759653334325	1.95841045146744	1.95839662903714	1.45258967743089e-07
0.7	2.08482976192656	2.08468092786128	2.08483038626034	2.08482977345274	1.15261804545241e-08
0.8	2.18012869628121	2.18057486442437	2.18011938738664	2.18012880646653	1.10185319979195e-07
0.9	2.25025936363533	2.25099079916225	2.25024731856312	2.25025948608599	1.22450661610429e-07
1	2.30009350539326	2.30076898469028	2.30008316906932	2.30009360749977	1.02106509913824e-07
$S.S.E_1 = 1.98649353681530e-13$					

Table2a : comparison between suggested method P_9 and other methods of example

t_i	I_1:exact	I_1 by using Rk method	I_1 by using AMB method	P_9 by using Osculatory interpolation	$\|I_1(exact)-P_9\|$
0	0	0	0	0	0
0.1	0.53826390677241	0.53826418298789	0.538265256576810	0.53826390466320	2.10921502485917e-09
0.2	0.96851299410465	0.96851316427637	0.968518612724965	0.96851295820640	3.58982520376472e-08
0.3	1.31073654702733	1.31073664052386	1.31075230563621	1.31073641467898	1.32348350279088e-07
0.4	1.58128435041602	1.58128438944319	1.58131037150845	1.58128411224080	2.38175224209769e-07
0.5	1.79352704806760	1.79352704930026	1.79356004425530	1.79352679445575	2.53611845080215e-07
0.6	1.95839677429611	1.95839675015366	1.95843377163562	1.95839662903714	1.45258967743089e-07
0.7	2.08482976192656	2.08482972156783	2.08486821414224	2.08482977345274	1.15261804545241e-08
0.8	2.18012869628121	2.18012864637841	2.18015813273662	2.18012880646653	1.10185319979195e-07
0.9	2.25025936363533	2.25025930897462	2.25028215152977	2.25025948608599	1.22450661610429e-07
1	2.30009350539326	2.30009340202130	2.30011239767724	2.30009360749977	1.02106509913824e-07

Table2b : Comparison between PP_9 and other methods of example

t_i	I_2:exact	I_2 by using Rk method	I_2 by using AMB method	PP_9 by using Osculatory interpolation	$\|I_2-PP_9\|$
0	0	0	0	0	0
0.1	0.31963204366726	0.31963222778619	0.319632943509476	0.31963204226148	1.40578487739518e-09
0.2	0.56879167578974	0.56879178921492	0.568795421414105	0.56879165185902	2.39307186289395e-08
0.3	0.76074480140204	0.76074486371240	0.760755306759394	0.76074471317313	8.82289146186821e-08
0.4	0.90633335591022	0.90633338190952	0.906350702589317	0.90633319713278	1.58777446968017e-07
0.5	1.01441545178971	1.01441545259441	1.01443744819859	1.01441528272485	1.69064866328839e-07
0.6	1.09222571059729	1.09222569448688	1.09225037412191	1.09222561377241	9.68248783372872e-08
0.7	1.14567024940677	1.14567022248702	1.14569588255824	1.14567025711131	7.70453856269171e-09
0.8	1.17956816792783	1.17956813464680	1.17958779069321	1.17956824141258	7.34847451777654e-08
0.9	1.19784923516125	1.19784919870962	1.19786442562600	1.19784931683205	8.16708030093594e-08
1	1.20371571629781	1.20371564739065	1.20372830974375	1.20371578441702	6.81192080520532e-08

5. Residual control for System of IVP

Suppose that a first order system of ODEs :

$$y'(x) = f(x, y(x)), \qquad 0 \le x \le 1$$

Consider method suggested in this paper that approximate the solution of y(x) of (3) on a mesh :

$0 = x_0 < x_1 < \ldots\ldots < x_N = 1$ by a function P(x) that is smooth on subinterval $[x_i, x_{i+1}]$. The mesh spacing $h_i = x_{i+1} - x_i$ and convergence is the considered as $h = \max_i h_i$, where we have $y_i = y(x_i)$ is the exact solution. The local solution P(x) is defined by :

$$P'(x) = f(x, P(x)), \quad P(x_i) = y_i .$$

In taking a step of size h from x_i to obtain y_{i+1}, the local error of a semi-analytic method is :

$$le_i = P(x_i + h) - y_{i+1} \qquad (15)$$

This error is estimated by taking the step with two formulas, one producing an approximation P_{i+1} with local error $O(h^{n+1})$ and another producing u_{i+1} with local error $O(h^{n+2})$. It is then immediate that

$$\mathbf{u_{i+1} - P_{i+1} = le_i + h.o.t.}$$

Here "h.o.t." is " higher order terms ". At each step adjust h_i so that a norm of the local error is no bigger than a given tolerance.

Suppose Osculatory polynomial function p(x) that approximates y(x) on all of [0, 1].

We proposes that the size of the defect (residual),

$$\mathbf{r(x) = p'(x) - f(x , p(x)) ,}$$

be controlled. The term r(x) is a perturbation of the given ODE and we ask whether this perturbation is small .

At each step we can evaluate the residual wherever we like and use these values to estimate the size of the residual over the step. Then :

$$r(x) = [\, p'(x) - u'(x) \,] + [\, f(x , u(x)) - f(x , p(x)) \,] \qquad (16)$$

With the usual assumption that f satisfies a Lipschitz condition :

$$\mathbf{\left| \; f(x , P(x)) - f(x ,u(x)) \; \right| \leq L \left| P(x) - u(x) \right|}$$

the second term on the right hand of (16) is of the same order of accuracy as u(x)−p(x) .Generally p(x) approximates u(x) to higher order than p′(x) approximates u′(x) , i.e., assume that e(x) = P(x) – u(x) is $O(h^{n+1})$ and P'(x) – u'(x) is $O(h^n)$. Assuming that :

$$\mathbf{r(x) = -\,[\, u'(x) - p'(x) \,] + h.o.t.}$$

In the course of taking a step, semi-analytic method produce approximations to y(t) at a number of points in $[\, x_i , x_i + h \,]$.

This implies that the scaled residual is $O(h^{n+1})$, the same as the local error. Now , if the residuals are uniformly small, P(x) is a good solution in the sense that it is the exact solution of a problem close to the one supplied to the solver .

Now , prove there is a more useful connection between scaled residual and local error. To investigate this relationship ,we begin by integrating (16) over a subinterval of $[x_i , \beta]$, where $\beta \in (x_i , x_{i+1}]$, we get :

$$\int_{xi}^{\beta} r(x)\, dx = e(\beta) - e(x_i) - \int_{xi}^{\beta} [\, f(x , P(x)) - f(x, u(x) \,]\, dx \qquad (17)$$

Suppose now that the method of order n is super convergent at mesh point , meaning that if the method is of order n , a norm of the error at mesh points is at least $O(h^{n+1})$ [10] , so that $e(x_i)$ is $O(h^{n+1})$ and $\| e(\beta) \| = \| e(x) \|_i$. As argued earlier , the integrand on the right hand side of (17) is $O(h^{n+1})$ and the interval is of length no bigger than h , so $\int_{xi}^{B} \| \mathbf{r(x)\, dx} \| = \| \mathbf{e(x)} \|_{i.} + \mathbf{O(h^{n+1})}$

Then we have the inequality :

$$\int_{xi}^{B} \| \mathbf{r(x)\, dx} \| \leq \mathbf{h} \;\; \| \mathbf{r(x)} \|_{i.}$$

Then the size of the scaled residual is an upper bound on the size of the local error. Now , if we require that $\mathbf{max_i\, h \,\|r(x)\|_{i.} \leq \epsilon}$, for a tolerance ϵ , then we have $\mathbf{max_i \,\|e(x)\|_{i.} \leq \epsilon}$, this is a strong argument for controlling the size of the residual .

6. Conclusions

A remarkable advantage of the repeated interpolation technique for solving system of first order ordinary differential equation with initial conditions is that it is easily implemented and gives a result with high accuracy. The high accuracy of the method is confirmed by example and the suggested method compared with conventional methods via example and is shown to be that seems to converge faster and more accurately than the conventional methods .

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