COMPLEMENTED SEMIRINGS

C. Venkata Lakshi*1, T. Vasanthi*2

#1 Dept. of Applied Mathematics, Sri Padmavathi Mahila Visva Vidyalayam, Tirupati –2 (A.P), India.

#2 Dept. of Applied Mathematics, Yogi Vemana University, Kadapa – 516003(A.P), India

ABSTRACT

In this paper, we study the properties of complemented semirings and totally ordered complemented semirings. This paper contains two sections. In section 1, we characterize complemented semirings. Here we proved that in a semiring \((S, +, \cdot)\), if ‘a’ is a complemented element then \(a^n + b^n = 1\), for all \(n \geq 1\). In section 2, we study the properties of totally ordered complemented semirings. Here we established that in a totally ordered.

Key words: Band; Minimum (Maximum) element; Positively totally ordered (p.t.o.) semigroup; Negatively totally ordered (n.t.o.) semigroup.

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Corresponding Author: C. Venkata Lakshi

INTRODUCTION

Semiring theory stands with a foot in each of two mathematical domains. On one hand, semirings are abstract mathematical structures and their study is part of abstract algebra- arising ab initio from the work of Dedekind, Macaulay, Krull and others on the theory of ideals of a commutative ring and then through the more general work of Vandon- and the tools used to study them is primarily the tools of abstract algebra. On the other, the modern interest in semirings arises primarily from fields of Applied Mathematics such as Optimization theory, the theory of discrete-event dynamical systems, automata theory and formal language theory, as well as from the allied areas of theoretical computer science and theoretical physics and the questions being asked is, for the most part, motivated by applications.

During the last three decades, there is considerable impact of semigroup theory and semiring theory on the development of ordered semirings both in theory and applications, which are akin to ordered rings and ordered semirings. In this direction the works of H.J. Weinert, M. Satyanarayana [1], J. Hanumanthachari, T.Vasanthi [5], and H.J. Weinert, M. satyanarayana, J. Hanumanthachari and D. Umamaheswara Reddy [4], K.P. Shum and C.S. Hoo, K.P. Shum and C.Y. Hung, Kehayopulu, G. Therrin, H. Jurgensen, H.J. Shyr, Gerard Lallment, U. Zimmermann and Jonathan S. Golan[2] are to be worth mentioning.
The theory of semirings and ordered semirings find wide applications in linear and combinatorial optimization problems such as path problems, transportation and assignment problems, matching problems, and Eigen value problems.

In this paper, we investigate the additive and multiplicative properties of Complemented Semirings and ordered complemented Semirings.

**DEFINITION:**
A triple \((S, +, \cdot)\) is said to be a semiring if \(S\) is a non-empty set and \(\{+\), \(\cdot\}\) are binary operations on \(S\) satisfying that

(i) \((S, +)\) is a semigroup

(ii) \((S, \cdot)\) is a semigroup

(iii) \(a(b + c) = ab + ac\) and \((b + c)a = ba + ca\), for all \(a, b, c\) in \(S\).

**DEFINITION:**
A semiring \((S, +, \cdot)\) is said to be totally ordered semiring (t.o.s.r.) if there exists a partially order \(\leq\) on \(S\) such that

(i) \((S, +)\) is a t. o. s. g.

(ii) \((S, \cdot)\) is a t. o. s. g.

It is usually denoted by \((S, +, \cdot, \leq)\).

**DEFINITION:**
In a totally ordered semiring \((S, +, \cdot, \leq)\)

(i) \((S, +, \leq)\) is positively totally ordered (p.t.o.), if \(a + b \geq a, b\) for all \(a, b\) in \(S\) and

(ii) \((S, \cdot, \leq)\) is positively totally ordered (p.t.o.), if \(ab \geq a, b\) for all \(a, b\) in \(S\).

**DEFINITION:**
In a totally ordered semiring \((S, +, \cdot, \leq)\)

(i) \((S, +, \leq)\) is negatively totally ordered (n.t.o.), if \(a + b \leq a, b\) for all \(a, b\) in \(S\) and

(ii) \((S, \cdot, \leq)\) is negatively totally ordered (n.t.o.), if \(ab \leq a, b\) for all \(a, b\) in \(S\).

The concept of Complemented Semiring is taken from the book of Jonathan S. Golan [3], entitled “Semirings and Affine Equations over Them: Theory and Applications”.

1. **COMPLEMENTED SEMIRINGS**

In this section, the properties of complemented semirings are studied. We proved that if \(\text{‘}a\text{’}\) is a complemented element in a semiring, then \(a^n + b^n = 1\), for all \(n \geq 1\).

**DEFINITION 1.1:**
An element \(\text{‘}a\text{’}\) of a semiring \(S\) is complemented if and only if there exists an element \(b\) of \(S\) satisfying \(a + b = 1\) and \(ab = ba = 0\).

**DEFINITION 1.2:**
An element \(x\) in a semigroup \((S, \cdot)\) is said to be multiplicative idempotent if \(x^2 = x\). A semigroup \((S, \cdot)\) is said to be a band if every element in \(S\) is an idempotent.

**Note:**
\(E(\cdot)\) denotes the set of all multiplicative idempotents in \((S, \cdot)\).
\(|E(\cdot)|\) denotes the cardinal number of the set \(E(\cdot)\).
THEOREM 1.3: Let \((S, +, \cdot)\) is a complemented semiring containing the multiplicative identity 1. If \(S\) contains additive identity zero, then

(i) \((S, \cdot)\) is a band

(ii) \((S, +)\) is commutative

**PROOF:** Given that \((S, +, \cdot)\) is a complemented semiring

Therefore \(a + b = 1\) and \(ab = ba = 0\), for all \(a, b\) in \(S\)

(i) Consider \(a.1 = a.(a + b)\)

\[ \Rightarrow a = a^2 + ab \]

\[ \Rightarrow a = a^2 + 0 \quad (\because \ ab = 0) \]

\[ \Rightarrow a = a^2 \]

\(\therefore (S, \cdot)\) is a band.

(ii) Consider \(ab + b^2 + a^2 + ba = (a + b) b + (a + b) a\)

\[ = (a + b) (b + a) \]

\[ = a (b + a) + b (b + a) \]

\[ \Rightarrow 0 + b^2 + a^2 + 0 = 0 + a^2 + b^2 + 0 \quad (\because ab = 0 \text{ and } ba = 0) \]

\[ \Rightarrow b^2 + a^2 = a^2 + b^2 \]

\[ \Rightarrow b + a = a + b \quad (\because \text{from (i), } (S, \cdot) \text{ is a band, } a^2 = 0 \text{ and } b^2 = 0) \]

\(\therefore (S, +)\) is commutative.

This is evident from the following example

**EXAMPLE 1.4:**

\[
\begin{array}{ccc|cc}
+ & 0 & 1 & a & b \\
0 & 0 & 1 & a & b \\
1 & 1 & 1 & 1 & 1 \\
a & a & 1 & 1 & 1 \\
b & b & 1 & 1 & b \\
\end{array}
\]

\[
\begin{array}{ccc|cc}
\cdot & 0 & 1 & a & b \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & a & b \\
a & 0 & a & a & 0 \\
b & 0 & b & 0 & b \\
\end{array}
\]

THEOREM 1.5: Let \(\text{`a’}\) is a complemented element in a semiring. Then \(a^n + b^n = 1\), for all \(n \geq 1\).

**PROOF:** Since \(\text{`a’}\) is a complemented element, there exists some \(b \in S\), such that \(a + b = 1\)

\[ \Rightarrow (a + b)^2 = 1 \]

\[ \Rightarrow (a + b) (a + b) = 1 \]

\[ \Rightarrow a^2 + ab + ba + b^2 = 1 \quad (\because \text{using distributive laws}) \]

\[ \Rightarrow a^2 + 0 + 0 + b^2 = 1 \quad (\because \ ab = 0 \text{ and } ba = 0) \]

\[ \Rightarrow a^2 + b^2 = 1 \]

Now \((a + b)^3 = 1\)

\[ \Rightarrow a^3 + a^2 b + b^2 a + b^3 = 1 \]

\[ \Rightarrow a^3 + a.b + b.a + b^3 = 1 \]

\[ \Rightarrow a^3 + 0 + 0 + b^3 = 1 \quad (\because \ ab = 0 \text{ and } ba = 0) \]

\[ \Rightarrow a^3 + b^3 = 1 \]

Also \((a + b)^4 = 1\)

\[ \Rightarrow (a^2 + b^2) (a^2 + b^2) = 1 \]

\[ \Rightarrow a^4 + a^2 b^2 + b^2 a^2 + b^4 = 1 \]
\[ a^4 + a \cdot a \cdot b \cdot b + b \cdot a \cdot b + b^4 = 1 \]
\[ a^4 + 0 + 0 + b^4 = 1 \quad (\because ab = 0 \text{ and } ba = 0) \]
\[ a^4 + b^4 = 1 \]

Continuing like this, we get \( a^n + b^n = 1 \), for all \( n \geq 1 \).

2. TOTALLY ORDERED COMPLEMENTED SEMIRINGS

In this section, the properties of totally ordered complemented semirings are studied. We proved that in a totally ordered complemented semiring \((S, +, \cdot)\), if \((S, \cdot)\) is positively totally ordered \((p.t.o.)\), then \((S, +)\) is a band.

**THEOREM 2.1:** Let \((S, +, \cdot)\) be a totally ordered complemented semiring containing multiplicative identity 1. If \((S, \cdot)\) is positively totally ordered \((p.t.o.)\), then \((S, +)\) is negatively totally ordered \((n.t.o.)\).

**PROOF:** By hypothesis, \((S, +, \cdot)\) is a complemented semiring

We have \(a + b = 1\) and \(ab = ba = 0\), for all \(a, b\) in \(S\)

Since \((S, \cdot)\) is p.t.o., \(a.1 \geq a, 1\)

\[ a \geq 1 \]

\[ 1 \text{ is the minimum element} \]

\[ a + b = 1 \leq a, b \]

\[ a + b \leq a, b \]

\[ \therefore (S, +) \text{ is negatively totally ordered (n.t.o.).} \]

**THEOREM 2.2:** Let \((S, +, \cdot)\) be a totally ordered complemented semiring. If \((S, +)\) is positively totally ordered \((p.t.o.)\), then \((S, \cdot)\) is negatively totally ordered \((n.t.o.)\).

**PROOF:** Given that \((S, +, \cdot)\) is a totally ordered complemented semiring

\[ a + b = 1 \text{ and } ab = ba = 0, \text{ for all } a, b \text{ in } S \]

Since \((S, +)\) is p.t.o., 0 is the minimum element

\[ ab = 0 \leq a, b \]

\[ ab \leq a, b \]

\[ \therefore (S, \cdot) \text{ is negatively totally ordered (n.t.o.).} \]

**DEFINITION 2.3:**
An element ‘x’ in a semigroup \((S, +)\) is said to be a band if \(x + x = x\).

**Note:**
\(E (+)\) denotes the set of all additive idempotents in \((S, +)\).
\(|E (+)|\) denotes the cardinal number of the set \(E [+]\).

**THEOREM 2.4:** Let \((S, +, \cdot)\) be a totally ordered complemented semiring. If \((S, \cdot)\) is positively totally ordered \((p.t.o.)\), then \((S, +)\) is a band.

**PROOF:** Since \((S, \cdot)\) is positively totally ordered \((p.t.o.)\), 1 is the minimum element

So \(1 + 1 \geq 1\)

\[ x.(1 + 1) \geq x.1 \]

\[ x + x \geq x \quad \rightarrow (1) \]

Since \(x + y = 1 \leq x, y \) \(\therefore 1 \text{ is the minimum element} \)
\[ x + x = 1 \leq x \rightarrow (2) \]

\[ x + x = x \]

Hence \((S, +)\) is a band.

This is evident from the following example

**EXAMPLE 2.5:** \(1 < 0\)

\[
\begin{array}{c|cc}
+ & 1 & 0 \\
\hline
1 & 1 & 1 \\
0 & 1 & 0
\end{array}
\]

\[
\begin{array}{c|cc}
\cdot & 1 & 0 \\
\hline
1 & 1 & 0 \\
0 & 0 & 0
\end{array}
\]

**CONCLUSION**

1. In a complemented semiring \((S, +, \cdot)\) contains the multiplicative identity 1 and the additive identity zero then the multiplicative structure satisfies the band property and the additive structure becomes commutative.
2. A semiring \((S, +, \cdot)\) contains the complemented element ‘a’ then \(a^n + b^n\) becomes 1, for all \(n \geq 1\).
3. In a totally ordered complemented semiring \((S, +, \cdot)\), in which the additive structure is positively totally ordered then the multiplicative structure becomes negatively totally ordered.
4. In a totally ordered complemented semiring \((S, +, \cdot)\), if \((S, \cdot)\) is positively totally ordered then the additive structure satisfies the band property.

**REFERENCES**