## A higher order Finite difference method to solve the Poisson equation in a rectangular domain

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#### Abstract

In order to illustrate the numerical solution of the Poisson equation we consider the distribution of temperature in a two-dimensional, rectangular plate, where the temperature is maintained at given values along the four boundaries to the plate. We will present a higher order finite difference method to solve the Poisson equation using five and nine point approximation.


Keywords: Poisson equation, Finite difference method.

## 1 Introduction

Poisson Equations is a second order partial differential equation (PDE) that appears in many areas of science an engineering, such as electricity, fluid flow, and steady heat conduction. Poisson equation governs a variety of equilibrium physical phenomena such as temperature distribution in solids, electrostatics, inviscid and irrotational two-dimensional flow (potential flow), and groundwater flow. In order to illustrate the numerical solution of the Poisson equation, we consider the distribution of temperature in a two-dimensional, rectangular plate, where the temperature is maintained at given values along the four boundaries to the plate (i.e., Dirichlet-type boundary conditions). Solution of this equation, in a domain, requires the specification of certain conditions that the unknown function must satisfy at the boundary of the domain. When the function itself is specified on a part of the boundary, we call that part the Dirichlet boundary; when the normal

[^0]derivative of the function is specified on a part of the boundary, we call that part the Neumann boundary. In a problem, the entire boundary can be Dirichlet or a part of the boundary can be Dirichlet and the rest Neumann. A problem with Neumann condition specified on the entire boundary does not have a unique solution. In some problems, a linear combination of the function and its normal derivative is specified; such situations are called Robin or Fourier boundary condition. We will not deal with the Robin problem or Neumann boundary condition, but it is fairly straightforward to extend the method described here to these problems. We will not discuss the Neumman problem in this paper, it's will be presented in a second paper using the same technique. Finite difference methods for partial differential equations are studied in $[1,2,3,4,5]$. Idea of finite difference method is to descretize the partial differential equation by replacing partial derivatives with their approximation that is finite differences. In this method, the PDE is converted into a set of linear, simultaneous equations. Which are written in the matrix equation and then solution is obtained by solving the matrix equation or solution can be obtained by solving simultaneous equations iteratively. The purpose of this paper is to illustrate how to solve linear elliptic PDEs using high order finite difference method. For concreteness, we will focus on the following PDE:
$$
\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}=f(x, y)
$$
where the source function $f(x, y)$ is given. We will seek a solution on a rectangular region of the $x y$-plane: $(x, y) \in[-L, L] \times[-L, L]$ subject to Dirichlet boundary conditions:
$$
u(x, y=-L)=g_{1}(x), u(x=L, y)=g_{2}(y), u(x, y=+L)=g_{3}(x), u(x=-L, y)=g_{4}(y)
$$

Here, the functions $g_{i}, i=1, \ldots, 4$ are assumed to be given. If $f(x, y)=0$ this is known as Poisson's equation, if not it is Poisson's equation. The goal of the numerical analysis will to be to "fill-in" the values of $u(x, y)$ interior to the boundary using a accurate Finite Difference Method. This article is organized as follows: In section, 2, we present a finite different method using five point to solve the Poisson equation. In section 3, we give a configuration of nine pooint method used to solve the Poisson equation.

## 2 Five point Method

The key step in solving our PDE numerically using finite difference methods is to replace the derivatives with so-called "finite difference Method". Here is an example of the a centered finite difference method for the second derivative:

$$
\frac{\partial^{2}}{\partial x^{2}} u(x, y)=\sum_{i=-2}^{2} \beta_{i} u(x+i h, y)+\text { Error }
$$

The method is "centered" because $u$ is evaluated at an equal number of points to the right and left of the point where we want to approximate the derivative. The coefficients $\beta_{i}$ are specified such that Error is $O\left(h^{6}\right)$. That is, we solve for Error, expand in a Taylor series in $h$ and then we choose the coefficients such that the first 5 terms in the Taylor series vanish,i.e. Error $=O\left(h^{5}\right)$ if and only if:

$$
\left\{\begin{array}{r}
2 \beta_{-2}-\beta_{1}+\beta_{-1}-2 \beta_{2}=0 \\
1 / 6 \beta_{-1}-4 / 3 \beta_{2}+4 / 3 \beta_{-2}-1 / 6 \beta_{1}=0 \\
-1 / 24 \beta_{1}-1 / 24 \beta_{-1}-2 / 3 \beta_{2}-2 / 3 \beta_{-2}=0 \\
-2 \beta_{2}-2 \beta_{-2}-1 / 2 \beta_{1}-1 / 2 \beta_{-1}=0 \\
-\beta_{2}-\beta_{-2}-\beta_{1}-\beta_{-1}-\beta_{0}=0
\end{array}\right.
$$

this lead to:

$$
\beta_{-2}=-\frac{1}{12} h^{-2}, \beta_{-1}=\frac{4}{3} h^{-2}, \beta_{0}=-\frac{5}{2} h^{-2}, \beta_{1}=\frac{4}{3} h^{-2}, \beta_{2}=-\frac{1}{12} h^{-2}
$$

Now, we get the approximation of the second derivative:

$$
\begin{align*}
\frac{\partial^{2}}{\partial x^{2}} u(x, y)= & -\frac{1}{12} \frac{u(x-2 h, y)}{h^{2}}+\frac{4}{3} \frac{u(x-h, y)}{h^{2}}-\frac{5}{2} \frac{u(x, y)}{h^{2}} \\
& +\frac{4}{3} \frac{u(x+h, y)}{h^{2}}-\frac{1}{12} \frac{u(x+2 h, y)}{h^{2}}+O\left(h^{5}\right) \tag{1}
\end{align*}
$$

the same is done for the second term in Poisson equation:

$$
\begin{align*}
\frac{\partial^{2}}{\partial y^{2}} u(x, y)= & -\frac{1}{12} \frac{u(x, y-2 h)}{h^{2}}+\frac{4}{3} \frac{u(x, y-h)}{h^{2}}-\frac{5}{2} \frac{u(x, y)}{h^{2}} \\
& +\frac{4}{3} \frac{u(x, y+h)}{h^{2}}-\frac{1}{12} \frac{u(x, y+2 h)}{h^{2}}+O\left(h^{5}\right) \tag{2}
\end{align*}
$$

Notice that we have assumed the same stepsize $h$ in the $x$ and $y$ directions. Putting (1) and (2) in the Poisson equation we get:

$$
\begin{aligned}
& -\frac{1}{12} \frac{u(x-2 h, y)}{h^{2}}+\frac{4}{3} \frac{u(x-h, y)}{h^{2}}-5 \frac{u(x, y)}{h^{2}}+\frac{4}{3} \frac{u(x+h, y)}{h^{2}} \\
& -\frac{1}{12} \frac{u(x+2 h, y)}{h^{2}}-\frac{1}{12} \frac{u(x, y-2 h)}{h^{2}}+\frac{4}{3} \frac{u(x, y, h)}{h^{2}}+\frac{4}{3} \frac{u(x, y+h)}{h^{2}}-\frac{1}{12} \frac{u(x, y+2 h)}{h^{2}}=f(x, y)
\end{aligned}
$$

The first five terms are called the "five-point" Method of the Laplacian operator in 2 dimensional since it involves evaluation of $u(x, y)$ at five different points. Here is a sketch of the relative orientation of these points, see figure 2 .


Figure 1: Five point Method of the Laplacian

The error in the method by expanding in a Taylor series and then making use of the original PDE:

$$
\left(\left(-\frac{1}{90}\left(D_{1,1,1,1,1,1}\right)(u)(x, y)-\frac{1}{90}\left(D_{2,2,2,2,2,2}\right)(u)(x, y)\right) h^{4}+O\left(h^{6}\right)\right)
$$

It is common in the literature to conclude from this result that the error in the method is $O\left(h^{4}\right)$. On the $u$ terms all being divided by $h^{2}$, we had instead written our method as

$$
\begin{aligned}
& -\frac{1}{12} u(x-2 h, y)+\frac{4}{3} u(x-h, y)-5 u(x, y)+\frac{4}{3} u(x+h, y)-\frac{4}{3} u(x+2 h, y) \\
& -\frac{1}{12} u(x, y-2 h)+\frac{4}{3} u(x, y-h)+\frac{4}{3} u(x, y+h)-\frac{1}{12} u(x, y+2 h)=h^{2} f(x, y)
\end{aligned}
$$

and expanded about $h=0$ the leading order behaviour would have been $O\left(h^{6}\right)$.

Having now obtained a discrete form of the PDE, we now turn our attention to how to exploit it and obtain a numeric solution. We first need to discretize the domain in ${ }^{\text {' }}(x, y) \in$ $[a, b] \times[c, d])$ over which we seek a solution. We will assume $N+2$ lattice points in the $x$ and $y$ directions, respectively. The coordinates of the lattice point will be explicitly given by $x_{i}=Z_{i}$ and $y_{j}=Z_{j}$ where $Z(i)=-L+\frac{2 i L}{N+1}, i=0, \ldots, N+1$. Here is a visualization of the lattice in the case $N=4$ and $L=1$, see Figure 3:


Figure 2: Visualization of the lattice in the case $N=4$ and $L=1$

In Figure 2, each node is labelled by our approximation to the true solution of the PDE $u_{i, j} \simeq u\left(x_{i}, y_{j}\right)$. The Boundary conditions will be used to fix $u_{i, j}$ at each of the purple boundary nodes, so we will solve for $u_{i, j}$ at each of the white interior nodes. We also define $f_{i, j}=f\left(x_{i}, y_{j}\right)$, which allows us to re-write the method as:

$$
\begin{aligned}
& -\frac{1}{12} \quad \frac{u_{i-2, j}}{h^{2}}+4 / 3 \frac{u_{i-1, j}}{h^{2}}-5 \frac{u_{i, j}}{h^{2}}+\frac{4}{3} \frac{u_{i+1, j}}{h^{2}} \\
& -\frac{1}{12} \quad \frac{u_{i+2, j}}{h^{2}}-\frac{1}{12} \frac{u_{i, j-2}}{h^{2}}+\frac{4}{3} \frac{u_{i, j-1}}{h^{2}}+\frac{4}{3} \frac{u_{i, j+1}}{h^{2}}-\frac{1}{12} \frac{u_{i, j+2}}{h^{2}}=f_{i, j}
\end{aligned}
$$

We will compute the solution $u_{i, j}, i=1, \ldots, N, j=1, \ldots, N$ of our PDE in the interior point of the lattice. The boundary conditions $u(x,-L)=g_{1}(x)$ etc are implemented by assigning values to all the boundary nodes (purple diamonds in the above sketch). For
example, we set, $u_{0, i}=g_{1}\left(x_{i}\right)$. There is a potential conflict at the corners of our lattice where $(i, j)=(0,0),(N+1,0),(N+1, N+1),(0, N+1)$, if $g_{1}(L) \neq g_{2}(-L), g_{2}(-L) \neq$ $g_{3}(L), g_{3}(-L) \neq g_{4}(L)$ or $g_{4}(-L) \neq g_{1}(-L)$. Ideally, we should choose boundary data to ensure that this does not happen, but if there is an ambiguity we will (arbitrarily) assume that the top and bottom boundary data take precedence over the left and right boundary data. With $N=5$, we will solve a linear system with size $4 \times 4$, to obtained the value of $u(x, y)$ in the interior nodes of the lattice. For simplicity, we show here, the shape of the matrix in the case where, $N=3$ Here, we present simple case, where $N=3$, and the linear system obtained $\mathcal{A W}=\mathcal{F}$. It is more convenient to reshape them into a vector as indicated by this before and after plot

$$
\mathcal{A}=\left[\begin{array}{ccccccccc}
-\frac{5}{h^{2}} & \frac{4}{3 h^{2}} & -\frac{1}{12 h^{2}} & \frac{4}{3 h^{2}} & 0 & 0 & -\frac{1}{12 h^{2}} & 0 & 0 \\
\frac{4}{3 h^{2}} & -\frac{5}{h^{2}} & \frac{4}{3 h^{2}} & 0 & \frac{4}{3 h^{2}} & 0 & 0 & -\frac{1}{12 h^{2}} & 0 \\
-\frac{1}{12 h^{2}} & \frac{4}{3 h^{2}} & -\frac{5}{h^{2}} & 0 & 0 & \frac{4}{3 h^{2}} & 0 & 0 & -\frac{1}{12 h^{2}} \\
\frac{4}{3 h^{2}} & 0 & 0 & -\frac{5}{h^{2}} & \frac{4}{3 h^{2}} & -\frac{1}{12 h^{2}} & \frac{4}{3 h^{2}} & 0 & 0 \\
0 & \frac{4}{3 h^{2}} & 0 & \frac{4}{3 h^{2}} & -\frac{5}{h^{2}} & \frac{4}{3 h^{2}} & 0 & \frac{4}{3 h^{2}} & 0 \\
0 & 0 & \frac{4}{3 h^{2}} & -\frac{1}{12 h^{2}} & \frac{4}{3 h^{2}} & -\frac{5}{h^{2}} & 0 & 0 & \frac{4}{3 h^{2}} \\
-\frac{1}{12 h^{2}} & 0 & 0 & \frac{4}{3 h^{2}} & 0 & 0 & -\frac{5}{h^{2}} & \frac{4}{3 h^{2}} & -\frac{1}{12 h^{2}} \\
0 & -\frac{1}{12 h^{2}} & 0 & 0 & \frac{4}{3 h^{2}} & 0 & \frac{4}{3 h^{2}} & -\frac{5}{h^{2}} & \frac{4}{3 h^{2}} \\
0 & 0 & -\frac{1}{12 h^{2}} & 0 & 0 & \frac{4}{3 h^{2}} & -\frac{1}{12 h^{2}} & \frac{4}{3 h^{2}} & -\frac{5}{h^{2}}
\end{array}\right]
$$

and

$$
\mathcal{F}=\left[\begin{array}{c}
1 / 12 \frac{u_{-1,1}}{h^{2}}-4 / 3 \frac{g_{4}\left(y_{1}\right)}{h^{2}}-4 / 3 \frac{g_{1}\left(x_{1}\right)}{h^{2}}+1 / 12 \frac{u_{1,-1}}{h^{2}}+f_{1,1} \\
1 / 12 \frac{g_{4}\left(y_{1}\right)}{h^{2}}+1 / 12 \frac{u_{2,-1}}{h^{2}}-4 / 3 \frac{g_{1}\left(x_{2}\right)}{h^{2}}+1 / 12 \frac{g_{2}\left(y_{1}\right)}{h^{2}}+f_{2,1} \\
1 / 12 \frac{u_{5,1}}{h^{2}}+1 / 12 \frac{u_{3,-1}}{h^{2}}-4 / 3 \frac{g_{1}\left(x_{3}\right)}{h^{2}}-4 / 3 \frac{g_{2}\left(y_{1}\right)}{h^{2}}+f_{3,1} \\
1 / 12 \frac{u_{-1,2}}{h^{2}}-4 / 3 \frac{g_{4}\left(y_{2}\right)}{h^{2}}+1 / 12 \frac{g_{3}\left(x_{1}\right)}{h^{2}}+1 / 12 \frac{g_{1}\left(x_{1}\right)}{h^{2}}+f_{1,2} \\
1 / 12 \frac{g_{4}\left(y_{2}\right)}{h^{2}}+1 / 12 \frac{g_{1}\left(x_{2}\right)}{h^{2}}+1 / 12 \frac{g_{2}\left(y_{2}\right)}{h^{2}}+f_{2,2}+1 / 12 \frac{g_{3}\left(x_{2}\right)}{h^{2}} \\
1 / 12 \frac{u_{5,2}}{h^{2}}+1 / 12 \frac{g_{1}\left(x_{3}\right)}{h^{2}}-4 / 3 \frac{g_{2}\left(y_{2}\right)}{h^{2}}+1 / 12 \frac{g_{3}\left(x_{3}\right)}{h^{2}}+f_{3,2} \\
1 / 12 \frac{u_{-1,3}}{h^{2}}-4 / 3 \frac{g_{4}\left(y_{3}\right)}{h^{2}}-4 / 3 \frac{g_{3}\left(x_{1}\right)}{h^{2}}+1 / 12 \frac{u_{1,5}}{h^{2}}+f_{1,3} \\
1 / 12 \frac{g_{4}\left(y_{3}\right)}{h^{2}}+f_{2,3}-4 / 3 \frac{g_{3}\left(x_{2}\right)}{h^{2}}+1 / 12 \frac{g_{2}\left(y_{3}\right)}{h^{2}}+1 / 12 \frac{u_{2,5}}{h^{2}} \\
1 / 12 \frac{u_{5,3}}{h^{2}}+f_{3,3}-4 / 3 \frac{g_{2}\left(y_{3}\right)}{h^{2}}+1 / 12 \frac{u_{3,5}}{h^{2}}-4 / 3 \frac{g_{3}\left(x_{3}\right)}{h^{2}}
\end{array}\right]
$$

Here is an example of the output when the source function is set to zero $f(x, y)=$ $\sin (x y)$ and $g_{1}=g_{2}=g_{3}=0$ and $g_{4}(x, y)=(2-x)(2+x)$.


Figure 3: Solution of Poisson equation with $N=10$

A second example can be proposed for the Poisson equation: $g(x, y)=\sin (\pi x) \cos (2 \pi y)$ with $f(x, y)=-5 \pi^{2} \sin (\pi x) \cos (2 \pi y)$.

The solution using the five point method is presented in the following figure:


Figure 4: Numerical solution(left figure) and Error curve with $N=50$

## 3 Nine-point Method

Here, we present a more accurate method to solve the Poisson equation. We will use a nine point method for the Laplacian of the form:

$$
\begin{align*}
\frac{\partial^{2}}{\partial x^{2}} u(x, y)+\frac{\partial^{2}}{\partial y^{2}} u(x, y) & =\frac{a_{-1,-1} u(x-h, y-h)}{h^{2}}+\frac{a_{0,-1} u(x, y-h)}{h^{2}}+\frac{a_{1,-1} u(x+h, y-h)}{h^{2}} \\
& +\frac{a_{-1,0} u(x-h, y)}{h^{2}}+\frac{a_{0,0} u(x, y)}{h^{2}}+\frac{a_{1,0} u(x+h, y)}{h^{2}}+\frac{a_{-1,1} u(x-h, y+h)}{h^{2}} \\
& +\frac{a_{0,1} u(x, y+h)}{h^{2}}+\frac{a_{1,1} u(x+h, y+h)}{h^{2}} \tag{3}
\end{align*}
$$

This method involves a square array of points centered about $(x, y)$ as shown in the plot:


Figure 5: Lattice configuration of nine point method

We will now calculate the error in the method, simplify the expression using the PDE, and then select the $a_{i, j}$ coefficients to minimize the error. More specifically, we'll try to eliminate the derivatives of $u(x, y)$ of order 2 and higher from the Taylor series of (3) using the following relations:

$$
\begin{aligned}
\frac{\partial^{2}}{\partial x^{2}} u(x, y) & =-\frac{\partial^{2}}{\partial y^{2}} u(x, y)+f(x, y) \\
\frac{\partial^{3}}{\partial x^{3}} u(x, y) & =-\frac{\partial^{3}}{\partial x \partial y^{2}} u(x, y)+\frac{\partial}{\partial x} f(x, y) \\
\frac{\partial^{4}}{\partial x^{4}} u(x, y) & =-\frac{\partial^{4}}{\partial x^{2} \partial y^{2}} u(x, y)+\frac{\partial^{2}}{\partial x^{2}} f(x, y)
\end{aligned}
$$

Here is our calculation of the error in our approximation in (3). The set of equations to be satisfied by the $a_{i, j}$ 's to ensure that the error is $O\left(h^{3}\right)$ or higher. This yield to a linear system of 14 equations for 9 unknowns; there is no solution. So, the method of approximation in (3) cannot be made to yield an error smaller that $O\left(h^{2}\right)$ when solving Poisson's equation. This is no better than the five point method of the previous section. However if we now specialize to Poisson's equation by setting $f(x, y)=0$ we can do a little better. We now have 8 equations for 9 unknowns, yielding a one parameter family of solutions for the coefficients:

$$
\begin{aligned}
& a_{-1,-1}=a_{-1,-1}, a_{-1,0}=4 a_{-1,-1}, a_{-1,1}=a_{-1,-1}, \\
& a_{0,-1}=4 a_{-1,-1}, a_{0,0}=-20 a_{-1,-1}, a_{0,1}=4 a_{-1,-1}, a_{1,-1}=a_{-1,-1}, \\
& a_{1,0}=4 a_{-1,-1}, a_{1,1}=a_{-1,-1}
\end{aligned}
$$

Here, we ask the question: What is the reason behind the different sizes of the system for Poisson problem and Poisson problem?. Basically, there are more indeterminants in the Poisson equation error from the source function resulting in more equations that must be satisfied to cancel all the terms of order $O\left(h^{2}\right)$ or less. The classic nine-point method is defined by the following assumption (of course, this is arbitrary and can be changed), $a_{1,1}=\frac{1}{6}$.

$$
\begin{aligned}
& a_{-1,-1}=1 / 6, a_{-1,0}=2 / 3, a_{-1,1}=1 / 6, a_{0,-1}=2 / 3 \\
& a_{0,0}=-10 / 3, a_{0,1}=2 / 3, a_{1,-1}=1 / 6, a_{1,0}=2 / 3, a_{1,1}=1 / 6
\end{aligned}
$$

We substitute this back into (3) we get:

$$
\begin{aligned}
\Delta u(x, y)-f(x, y) & =1 / 6 \frac{u_{i-1, j-1}}{h^{2}}+2 / 3 \frac{u_{i, j-1}}{h^{2}}+1 / 6 \frac{u_{i+1, j-1}}{h^{2}}+2 / 3 \frac{u_{i-1, j}}{h^{2}} \\
& -10 / 3 \frac{u_{i, j}}{h^{2}}+2 / 3 \frac{u_{i+1, j}}{h^{2}}+1 / 6 \frac{u_{i-1, j+1}}{h^{2}}+2 / 3 \frac{u_{i, j+1}}{h^{2}}+1 / 6 \frac{u_{i+1, j+1}}{h^{2}} \\
& -2 / 3 f_{i, j}-1 / 12 f_{i, j-1}-1 / 12 f_{i, j+1}-1 / 12 f_{i-1, j}-1 / 12 f_{i+1, j}
\end{aligned}
$$

Note that we have left the source term in this expression; the reason will be apparent shortly. Let's calculate the error with these specific values of the coefficients.

$$
\begin{equation*}
\text { Error(Poisson) }=\frac{1}{12} h^{2} \frac{\partial^{2}}{\partial x^{2}} f(x, y)+\frac{1}{12} h^{2} \frac{\partial^{2}}{\partial y^{2}} f(x, y)+O\left(h^{4}\right) \tag{4}
\end{equation*}
$$

We immediately see that the leading order term in the error is

$$
h^{2} \nabla^{2} f(x, y)=h^{4} \nabla^{4} u(x, y)
$$

so it will indeed vanish for Poisson's equation, and we will have a method with error $O\left(h^{4}\right)$. But the crucial thing to note is that $f(x, y)$ is a known function, so we can actually calculate the $h^{2}$ term in (4) explicitly. Hence, it is possible to modify our method in such a way as to cancel this error term. Let us compute:

$$
\begin{array}{ll}
\frac{1}{6} & \frac{u(x-h, y-h)}{h^{2}}+\frac{2}{3} \frac{u(x, y-h)}{h^{2}}+\frac{1}{6} \frac{u(x+h, y)}{h^{2}} \\
+\frac{2}{3} & \frac{u(x-h, y)}{h^{2}}-\frac{10}{3} \frac{u(x, y)}{h^{2}}+\frac{2}{3} \frac{u(x+h, y)}{h^{2}}+\frac{1}{6} \frac{u(x-h, y+h)}{h^{2}} \\
+\frac{2}{3} & \frac{u(x, y+h)}{h^{2}}+\frac{1}{6} \frac{u(x+h, y+h)}{h^{2}} \\
- & f(x, y)-\frac{1}{12} h^{2} \frac{\partial^{2}}{\partial x^{2}} f(x, y)-\frac{1}{12} h^{2} \frac{\partial^{2}}{\partial y^{2}} f(x, y) \tag{5}
\end{array}
$$

If the source is known analytically, we could in principle calculate the derivatives directly. But the source might not be known analytically (i.e., we could only have numeric knowledge), or we may be coding in an environment where we cannot take the derivative automatically. At any rate, we only need the derivatives to enough accuracy to negate the leading order term in (4), so we can use the followed centered method (it's of order two):

$$
\begin{align*}
\frac{\partial^{2}}{\partial x^{2}} f(x, y) & =\frac{f(x-h, y)}{h^{2}}-2 \frac{f(x, y)}{h^{2}}+\frac{f(x+h, y)}{h^{2}}  \tag{6}\\
\frac{\partial^{2}}{\partial y^{2}} f(x, y) & =\frac{f(x, y-h)}{h^{2}}-2 \frac{f(x, y)}{h^{2}}+\frac{f(x, y+h)}{h^{2}} \tag{7}
\end{align*}
$$

Subbing (6) in to (5) yield the final form of the nine-point method of the Poisson equation:

$$
\begin{align*}
\text { Nine-Point-method }:= & \frac{1}{6} \frac{u(x-h, y-h)}{h^{2}}+\frac{2}{3} \frac{u(x, y-h)}{h^{2}}+\frac{1}{6} \frac{u(x+h, y)}{h^{2}} \\
& +\frac{2}{3} \frac{u(x-h, y)}{h^{2}}-\frac{10}{3} \frac{u(x, y)}{h^{2}}+\frac{2}{3} \frac{u(x+h, y)}{h^{2}}+\frac{1}{6} \frac{u(x-h, y+h)}{h^{2}} \\
& +\frac{2}{3} \frac{u(x, y+h)}{h^{2}}+\frac{1}{6} \frac{u(x+h, y+h)}{h^{2}} \\
& -\frac{2}{3} f(x, y)-\frac{1}{12} f(x-h, y) \\
& -\frac{1}{12} f(x+h, y)-\frac{1}{12} f(x, y-h)-\frac{1}{12} f(x, y+h) \tag{8}
\end{align*}
$$

We confirm that the error in this nine point method is of order $O\left(h^{4}\right)$ for the Poisson equation (not just the Poisson equation):

$$
\begin{equation*}
\text { Error(Poisson) }=\frac{-1}{240} h^{4} \frac{\partial^{4}}{\partial x^{4}} f(x, y)+\frac{1}{90} h^{4} \frac{\partial^{4}}{\partial y^{4}} f(x, y)+O\left(h^{6}\right) \tag{9}
\end{equation*}
$$

Note that even this error is only a functional of $f(x, y)$, not of the solution $u(x, y)$. Hence, we could have even cancelled this term in the error by adding more terms to (8) if an even more accurate method is desired. However, the finite difference representation of the higher order derivatives in (9) will require more than the nine points already in (8) which is not desirable if we cannot take derivatives of analytically. Here is an example problem the calculates the electric potential around a ring of $2 n$ alternating charges. We model the charges as discs of radius $r$ located a distance of $R$ away from the origin and with surface charge density $\pm 1$. (For $n=1$, this is an electric dipole).

A simple example, we consider the case where $n:=2, r=4, R=7, L=25$ : Let then $g=0$,

$$
f(x, y)=\sum_{i=0}^{2 n-1}(-1)^{i} H\left(r^{2}-\left(x-R \cos \left(\frac{\pi i}{n}\right)\right)^{2}-\left(y-R \sin \left(\frac{\pi i}{n}\right)\right)^{2}\right)
$$

where $H$ is the Heaviside step function, or the unit step function:

$$
\forall x \in \mathbb{R}, H(x)=\left\{\begin{array}{lll}
0 & \text { si } & x<0 \\
1 & \text { si } & x \geq 0
\end{array}\right.
$$



Figure 6: Visualization of the function $f$ (left figure) and the solution $u(x, y)$ obtained using nine point method

## 4 Conclusion

Considering the results obtained in this paper, we plan in the future to tackle the following questions:

- Solve the Poisson equation with mixed boundary condition in a triangular domain.
- Using iterative method and higher order finite difference method to solve the Poisson/Poisson equation.
- Give a numerical solution of the Poisson equation in a non regular domain, let say a non convex domain.


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