On Estimation of Reliability Function of Consul and Geeta Distributions

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ABSTRACT

In this paper a modest attempt is made for estimating Bayesian Reliability of parametric function of θ, of Consul, Geeta and Size-biased Geeta distributions assuming β as known, and the prior distribution of parameter θ, is considered as two parameter Beta distribution In addition reliability functions of size-biased Geeta, Geometric, Negative-binomial and Haight distributions are also obtained.

Key words: Geeta Distribution, Consul Distribution, Beta Distribution, Reliability function.

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1. INTRODUCTION

Since the beginning of seventies, attention of research workers in discrete distribution appears to be shifted towards 'Lagrangian probability distributions'. The Lagrangian probability distributions provide generalization of classic discrete distributions that have been found more general in nature and wide in scope. The discovery of Lagrangian distribution has relaxed the research workers who were previously vexed in a large number of generalized mixture and compound discrete distributions as they have been found to have tremendous capability to fit well into observed distributions of any type. Consul and Shenton (1972) gave a method for generating new families of generalized discrete distributions with the help of Lagrangian expansion. Gupta (1974) defined Modified Power Series Distributions (MPSD) which is a sub class of Lagrangian probability distribution. The discrete Lagrangian probability distribution have been systematically studied in a number of papers by Consul and Shenton (1972.1973a,1973b,1975).

1.1) CONSUL DISTRIBUTION:

F.Famoye in (1997a) introduced the Consul distribution with parameters θ and m, where the probability mass function is given by

$$p(X=x)=\frac{1}{x}\binom{mx}{x-1}\theta^{x-1}(1-\theta)^{mx-x+1}\,; \quad x=1,2,3,\ldots \tag{1.1.1}$$

Where, $0 < \theta < 1 \quad and \quad 1 \leq m \leq \theta^{-1}$

The mean and variance of the model exists when $m < \theta^{-1}$. The mean and variance of (1.1.1) are given by the expressions

$$\mu = (1 - \theta m)^{-1} \tag{1.1.2}$$

and

$$\sigma^2 = m\theta(1-\theta)(1-\theta m)^{-3} \tag{1.1.3}$$

Using these values the Consul distribution can be expressed as a location-parameter discrete probability distribution in the form

$$p(X = x) = \begin{cases} \frac{m}{m+\beta x}\binom{m+\beta x}{x}\left(\frac{\mu-1}{m\mu}\right)^{x-1}\left(1-\frac{\mu-1}{m\mu}\right)^{m+\beta x-x} & ;\ x = 1,2,3,.. \\ 0 & ;\text{otherwise} \end{cases} \tag{1.1.4}$$

All the moments of the Consul distribution exist for $0 < \theta < 1 \quad and \quad 1 \leq m < \theta^{-1}$.

F. Famoye in (1997a) showed that the Consul distribution is the limit of zero-truncated Generalized Negative Binomial Distribution (GNBD)

$$P_x(\theta, \beta, m) = \frac{m}{m+\beta x}\binom{m+\beta x}{x}\theta^x(1-\theta)^{m+\beta x-x}/[1-(1-\theta)^m]; \quad x = 1,2,3,\ldots \tag{1.1.5}$$

As the parameter $\beta \to 1$ and is unimodal but not strongly unimodal for all values of $m \geq 1$ and $0 < \theta < 1$ and the mode is at a point $x = 1$. He also obtained moment estimates, the estimates based upon the sample mean and first frequency, and the maximum likelihood estimates. The model (1.1.1) is a member Lagrangian probability distribution and is L shaped. We now obtain the Bayesian estimators of a number of parameters functions of the parameter θ and the Bayesian Reliability function. Since, $0 < \theta < 1$, it is assumed that the prior information on θ may be summarized by a beta distribution, β(a, b) where the parameters 'a' and 'b' are not known.

1.2) GEETA DISTRIBUTION

Consul in (1990a) introduced the Geeta distribution, with parameters θ and β, where the probability mass function is defined

$$p(X = x) = \begin{cases} \frac{1}{\beta x - 1}\binom{\beta x - 1}{x}\theta^{x-1}(1-\theta)^{\beta x - 1} & ;\ x = 1,2,3\ldots \\ 0 & ;\ \text{elsewhere} \end{cases} \tag{1.2.1}$$

where $$0 < \theta < 1\,, 1 < \beta < \frac{1}{\theta}$$

The upper limit on β has been imposed for the existence of the mean of the distribution. When β→1, the Geeta distribution degenerates and its probability mass gets concentrates at point x = 1. Consul (1990a) studied the estimation of the model (1.2.1), using moments, sample mean and frequency, maximum likelihood estimation (MLE) and minimum variance unbiased estimation (MVUE) methods, and gave the MVUE estimates for some functions of parameter θ, which are similar to the results obtained by Gupta (1974) for modified power series distribution. Consul (1990b) gave two stochastic models for Geeta distribution and showed that the distribution can be obtained as an urn model and that it is also generated as a model based on a difference differential equation. The model (1.2.1) is a member of Consul and Shentons's (1972), Lagrangian probability distribution and also Gupta's (1974) modified power series distribution.
Geeta distribution is L- shaped (or reversed J-shaped) for all values of θ and β. The mean and variance of (1.2.1) are given by the expressions

$$\mu = \frac{(1-\theta)}{(1-\theta\beta)^{-1}} \tag{1.2.2}$$

and

$$\sigma^2 = (\beta-1)(1-\theta)(1-\theta\beta)^{-3} \tag{1.2.3}$$

The family of Geeta probability models belongs to the classes of the modified power series distribution and the Lagrangian series distribution. Consul (1990b) also expressed it as a location –parameter probability distribution given below:

$$p(X=x) = \begin{cases} \frac{1}{\beta x-1}\binom{\beta x-1}{x}\left(\frac{\mu-1}{\beta\mu-1}\right)^{x-1}\left(\frac{\mu(\beta-1)}{\beta\mu-1}\right)^{\beta x-x} & ; x = 1,2,3,\ldots \\ 0 & ; \text{otherewise} \end{cases} \tag{1.2.4}$$

A numerical approach indicates that Geeta distribution has maximum at x = 1, and can have a either short or long or heavy tail, depending upon the values of β and θ. Estimation using (1) moments (2) sample mean and first frequency, (3) M.L and (4) M.V.U.E are studied by Consul (1990a), two models of genesis (a two-urn model and a regenerative stochastic process) are given in Consul (1990b). Generating function and recurrence relations for central moments are given by Consul (1990a). We now obtain the Bayesian estimators of a number of parameters functions of the parameter θ and the Bayesian Reliability function. Since, $0 < \theta < 1$,it is assumed that the prior information on θ may be summarized by a beta distribution, β(a, b) where the parameter a and b are not known.

1.3) SIZE-BIASED GEETA DISTRIBUTION (SBGET)

The weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded; instead they are recorded according to some weight function. When the weight function depends on the lengths of the units of interest, the resulting distribution is called length biased. More generally, when the sampling mechanism selects units with probability proportional to some measure of the unit size, resulting distribution is called size-biased. Size-biased distributions are a special case of the more general form known as weighted distributions. The concept of weighted distribution can be traced to Fisher in his paper "The study of effect of methods of ascertainment upon estimation of frequencies" in 1934; while this of length-biased sampling was introduced by Cox 1962 (see Patill 2002

If $f(x;\theta)$ is the distribution of a random variable X with unknown parameter θ, then the corresponding size-biased distribution is of the form

$$f^*(x;\theta) = \frac{x^{\alpha} f(x;\theta)}{\mu'_{\alpha}}$$

where,
$$\mu'_{\alpha} = \sum x^{\alpha} f(x;\theta)$$

For $\alpha = 1$ and 2, we get the simple size-biased and area-biased distributions respectively. A Size-biased Geeta Distribution (SBGET) is obtained by applying the weight x^{α}, where $\alpha = 1$ to the Geeta distribution (1.2.1). This gives the size-biased Geeta distribution as

$$p_{sb}(X = x) = (1-\theta\beta)\binom{\beta x-2}{x-1}\theta^{x-1}(1-\theta)^{(\beta-1)x-1} \quad ; x = 1,2,3,\dots \qquad (1.3.1)$$

Where, $1 < \beta < \frac{1}{\theta}$ and $0 < \theta < 1$

2) BAYES ESTIMATOR (β KNOWN) OF RELIABILITY FUNCTION FOR CONSUL DISTRIBUTION

The likelihood function of Consul distribution is given by

$$L_1\left(\frac{x}{\theta},\beta\right) = k_1\theta^{z-n}(1-\theta)^{mz-z+n} \qquad (2.1)$$

where, $k_1 = \prod_{i=1}^{n}\left(\frac{1}{x}\binom{mx}{x-1}\right)$ and $z = \sum_{i=1}^{n} x_i$

Since 0<θ<1, it is assumed that prior information on θ is given by a beta distribution with density function

$$\lambda(\theta,a,b) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)} \quad ; a > 0, b > 0, 0 < \theta < 1 \qquad (2.2)$$

Thus, the joint p.d.f of $(X_1, X_2 \dots X_n, \theta)$ is given by

$$H\left(\frac{X_1, X_2, \dots, X_n}{\theta}\right) = L_1(x/\theta,\beta).\lambda(\theta,a,b)$$

then using Bayes theorem, the posterior distribution of θ becomes

$$p_1\left(\theta/_t\right) = \frac{\theta^{z+a-n-1}(1-\theta)^{(m-1)z+n+b-1}}{B(z+a-n,(m-1)z+n+b)} \tag{2.3}$$

Bayes estimator of any function $\varphi(\theta) = \theta^{x-1}(1-\theta)^{mx-x+1}$ with respect to squard error loss function is

$$\hat{\varphi}(\theta) = \frac{\int_0^1 \varphi(\theta)\theta^{z+a-n-1}(1-\theta)^{(m-1)z+n+b-1}\, d\theta}{B(z+a-n,(m-1)z+n+b)} \tag{2.4}$$

$$= \frac{B(z+a+x-n-1,(m-1)(x+z)+n+b+1)}{B(z+a-n,(m-1)z+n+b)} \tag{2.5}$$

Bayes estimator of Reliability function $R_c(t_0)$ for Consul distribution at a sepecified value $t_0(\geq 0)$ is

$$\hat{R}_c(t_0) = \sum_{x=t_0}^{\infty} \frac{1}{x}\binom{mx}{x-1} \frac{B(z+a+x-n-1,(m-1)(x+z)+n+b+1)}{B(z+a-n,(m-1)z+n+b)} \tag{2.6}$$

Let $x_1, x_2, x_3, \ldots, x_n$ are i.i.d Consul random variables as defined in (1.1.1), then the sample sum Y= $\sum x_i$ has Delta-binomial distribution given by

$$p_{db}(Y=y) = \begin{cases} \frac{n}{y}\binom{my}{y-n}\theta^{y-n}(1-\theta)^{my-y+n} & ; y = n, n+1, n+2, \ldots \\ 0 & ; \text{ elsewhere} \end{cases} \tag{2.7}$$

Using (2.4) we have Bayes estimate for $\theta^{y-n}(1-\theta)^{\beta y-1}$ is

$$\frac{B(z+a+x-n-1,(m-1)(x+z)+n+b+1)}{B(z+a-n,(m-1)z+n+b)} \tag{2.8}$$

Bayes estimator of Reliability, $R_{db}(t_0)$ at a sepecified value $t_0(\geq 0)$ is

$$\hat{R}_{db}(t_0) = \sum_{x=t_0}^{\infty} \frac{n}{y}\binom{my}{y-n} \frac{B(z+a+x-n-1,(m-1)(x+z)+n+b+1)}{B(z+a-n,(m-1)z+n+b)} \tag{2.9}$$

PARTICULAR CASE

1. When $m = 1$ (1.1.1) reduces to Geometric distribution with probability of success$1-\theta$. Then, the reliability function $\hat{R}(t_0)$ for Geometric distribution at time $t_0(\geq 0)$ is

$$\hat{R}_g(t_0) = \sum_{x=t_0}^{\infty} \frac{B(z+a+x-n-1,n+b+1)}{B(z+a-n,n+b)} \tag{2.10}$$

2. When $m = 1$ (2.7) reduces to negative binomial distribution with probability of success$1-\theta$. Then, the reliability function $R_{nb}(t_0)$ for negative binomial distribution at time $t_0 (\geq 0)$ is

$$\widehat{R}_{nb}(t_0) = \sum_{x=t_0}^{\infty} \binom{y-1}{n-1} \frac{B(z+a+x-n-1, n+b+1)}{B(z+a-n, n+b)} \tag{2.11}$$

3) BAYES ESTIMATOR (β KNOWN) OF RELIABILITY FUNCTION FOR GEETA DISTRIBUTION

The likelihood function of Geeta distribution is given by

$$L_2 (x/\theta, \beta) = k_2 \theta^{t-1} (1-\theta)^{\beta t - 1} \tag{3.1}$$

Where,

$$k_2 = \prod_{i=1}^{n} \frac{1}{\beta x_i - 1} \binom{\beta x_i - 1}{x_i} \quad \text{and} \quad t = \sum_{i=1}^{n} x_i$$

The part of the likelihood function which is relevant to Bayesian inference on unknown parameter θ is

$$\theta^{t-1} (1-\theta)^{\beta t - 1} \tag{3.2}$$

we assume that before the observations were made, our knowledge about θ was only vague, since 0<θ<1, it is assumed that prior information on θ is given by a beta distribution with density function

$$\lambda(\theta, a, b) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)} \;; a > 0, b > 0, 0 < \theta < 1 \tag{3.3}$$

Thus the joint p.d.f of $(X_1, X_2 \ldots X_n, \theta)$ is given by

$$H\left(\frac{X_1, X_2, \ldots, X_n}{\theta}\right) = L_2(x/\theta, \beta).\lambda(\theta, a, b)$$

then using Bayes theorem, the posterior distribution of θ becomes

$$p_2\left({}^{\theta}/_{t}\right) = \frac{\theta^{t+a-n-1}(1-\theta)^{(\beta-1)t+b-1}}{B(t+a-n, \beta(t-1)+b)} \tag{3.4}$$

Bayes estimator of any function $\varphi(\theta) = \theta^{x-1}(1-\theta)^{\beta x - 1}$ with respect to squard error loss function is

$$\hat{\varphi}(\theta) = \frac{\int_0^1 \varphi(\theta)\theta^{t+a-n-1}(1-\theta)^{(\beta-1)t+b-1}\ d\theta}{B(t+a-n, \beta(t-1)+b)} \tag{3.5}$$

$$= \frac{B(t+a+x-n-1, (\beta-1)(x+t)+b)}{B(t+a-n, \beta(t-1)+b)} \tag{3.6}$$

Also,
Table 1. Bayes estimators of some parameter function in Geeta distribution

$\varphi(\theta)$	$\hat{\varphi}(\theta)$
$\theta^l(1-\theta)^k$, l & k are non-negative integers	$\frac{B(n\bar{x}+a-n+1, (\beta-1)n\bar{x}+b+k)}{B(n\bar{x}+a-n, (\beta-1)n\bar{x}+b)}$
$\left[\theta\left(1-\theta^{\beta-1}\right)\right]^k$, k is positive integer	$= \frac{B(n\bar{x}+a-n+k, \beta n\bar{x}-n\bar{x}+b+k\beta-k)}{B(n\bar{x}+a-n, (\beta-1)n\bar{x}+b)}$
$p(X=k)$ $k=0,1,2,\ldots.$	$= \frac{1}{\beta k-1}\binom{\beta k-1}{x}\frac{B(n\bar{x}+a-n+k, \beta n\bar{x}-n\bar{x}+b+k\beta-k)}{B(n\bar{x}+a-n, (\beta-1)n\bar{x}+b)}$

Bayes estimator of Reliability, $R_{ge}(t_0)$ at a sepecified value $t_0(\geq 0)$ is

$$\hat{R}_{ge}(t_0) = \sum_{x=t_0}^{\infty} \frac{1}{\beta x-1}\binom{\beta x-1}{x}\frac{B(t+a+x-n-1, (\beta-1)(x+t)+b)}{B(t+a-n, \beta(t-1)+b)} \tag{3.7}$$

Let $x_1, x_2, x_3, \ldots, x_n$ are i.i.d Geeta random variables as defined in (1.2.1), then the sample sum Y= $\sum x_i$ has Geeta-n distribution given by

$$p(Y=y) = \frac{n}{y}\binom{\beta y-n-1}{y-n}\theta^{y-n}(1-\theta)^{\beta y-y} \quad ; y=n, n+1, n+2, \ldots \tag{3.8}$$

Using (3.5) we have Bayes estimate for $\theta^{y-n}(1-\theta)^{\beta y-1}$ is

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$$\frac{B(t+a+y-n-1,(\beta-1)(t+y)+b)}{B(t+a-n,(\beta-1)t+b} \tag{3.9}$$

Therefore, Bayes estimate of Reliability, $R_{gn}(t_0)$ at time $t_0(\geq 0)$ is

$$\widehat{R}_{gn}(t_0) = \sum_{y=t_0}^{\infty} \frac{n}{y}\binom{\beta y-n-1}{y-n}\frac{B(t+a+y-n-1,(\beta-1)(t+y)+b)}{B(t+a-n,(\beta-1)+b)} \tag{3.10}$$

PARTICULAR CASE

When $\beta = 2$, (3.8) reduces to Haight distribution. Therefore reliability function $\widehat{R}_h(t_0)$ for Haight distribution at time $t_0(\geq 0)$ is

$$\widehat{R}_h(t_0) = \sum_{y=t_0}^{\infty} \frac{n}{y}\binom{2y-n-1}{y-n}\frac{B(t+a+y-n-1,(t+y)+b)}{B(t+a-n,t+b)} \tag{3.11}$$

4) BAYES ESTIMATOR (β KNOWN) OF RELIABILITY FUNCTION FOR SIZE-BIASED GEETA DISTRIBUTION

The likelihood function of SBGET (1.3.1) is

$$L_3(x/\theta,\beta) = \left(\prod_{i=1}^{n}\binom{\beta x_i-2}{x_i-1}\right)(1-\theta\beta)^n\theta^{\sum x-n}(1-\theta)^{(\beta-1)\sum x-n}$$

$$L_3(x/\theta,\beta) = k_3(1-\theta\beta)^n\theta^{z-n}(1-\theta)^{(\beta-1)z-n} \tag{4.1}$$

Where,

$$k_3 = \prod_{i=n}^{n}\binom{\beta x_i-2}{x_i-1} \quad \text{and} \quad z = \sum_{i=1}^{n} x_i$$

Since, $0 < \theta < 1$, therefore we assume that the prior information about θ when β is know from Beta distribution , β(a, b) where the parameter a and b are not known.

$$\lambda(\theta,a,b) = \frac{\theta^{a-1}(1-\theta)^{b-1}}{B(a,b)} \;;\; a>0, b>0, 0<\theta<1 \tag{4.2}$$

The posterior distribution of θ from (4.1) and (4.2) is

$$p_3(\theta/t) = \frac{(1-\theta\beta)^n\theta^{z+a-n-1}(1-\theta)^{(\beta-1)z+b-n-1}}{\int_0^1(1-\theta\beta)^n\theta^{z+a-n-1}(1-\theta)^{(\beta-1)z+b-n-1}\,d\theta}$$

$$p_3(\theta/t) = \frac{(1-\theta\beta)^n \theta^{z+a-n-1}(1-\theta)^{(\beta-1)z+b-n-1}}{B(z+a-n,(\beta-1)z+b-n)\ {}^2F_1[-n,z+a-n;\beta z+a+b-2n;\beta]} \tag{4.3}$$

Bayes estimator of any function $\varphi(\theta) = (1-\beta\theta)\theta^{x-1}(1-\theta)^{(\beta-1)x-1}$ with respect to squard error loss function is

$$\hat{\varphi}(\theta) = \frac{\int_0^1 (1-\theta\beta)^{n+1}\,\theta^{z+a+x-n-2}(1-\theta)^{(\beta-1)(x+z)+b-n-2}d\theta}{B(z+a-n,(\beta-1)z+b-n)\ {}^2F_1[-n,z+a-n;\beta z+a+b-2n;\beta]}$$

$$\hat{\varphi}(\theta) = \frac{B(z+a+x-n-1,(\beta-1)(x+z)+b-n-1)\ {}^2F_1[-(n+1),z+a+x-n-1;\beta(x+z)+a+b-2(n+1);\beta]}{B(z+a-n,(\beta-1)z+b-n)\ {}^2F_1[-n,z+a-n;\beta z+a+b-2n;\beta]} \tag{4.4}$$

Therefore, Bayes estimator of Reliabity function $R_{sb}(t_0)$ for Size-biased Geeta distribution at a specified value $t_0\ (\geq 0)$ is

$$\hat{R}_{sb}(t_0) = \sum_{x=t_0}^{\infty}\binom{\beta x-2}{x-1} \times \frac{B(z+a+x-n-1,(\beta-1)(x+z)+b-n-1)\ {}^2F_1[-(n+1),z+a+x-n-1;\beta(x+z)+a+b-2(n+1);\beta]}{B(z+a-n,(\beta-1)z+b-n)\ {}^2F_1[-n,z+a-n;\beta z+a+b-2n;\beta]} \tag{4.5}$$

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