# Numerical solution of Volterra-Fredholm Integral Equations with Singular kernals using Toeplitz matrices.

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Abstract: This paper deals with an improvement of the numerical method based on Toeplitz matrices to solve the Volterra Fredholm Integral equation of the second kind with singular kernel. The kernel function $\mathcal{K}(s,t)$ is moderately smooth on $[a,b] \times [0,T]$ except possibly across the diagonal $s = t$. We transform the Volterra integral equations to a system of Fredholm integral equations of the second kind which will be solved by Toeplitz matrices method. This lead to a system of algebraic equations. Thus, by solving the matrix equation, the approximation solution is obtained. The accurate numerical solution will be presented. Keywords: Mixed integral equation, Toeplitz matrix method, logarithmic kernel, Carleman function.



## 1 Introduction

The mathematical physics and contact problems in the theory of elasticity lead to an integral equation of the first or second kind see [1, 2], Abdou et al. [3, 4, 5] discussed some different methods to solve FIE of the first kind with logarithmic kernel and Carleman function respectively. In [6], the author obtained the spectral relationships for the VFIE of the first kind. In [7, 8], the authors present some techniques to get analytic solution of the integral equation of the first and second kind, also some numerical methods are discussed in [9, 10]. In this paper, we consider the VFIE of the first kind:

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$$\int_0^t \int_\Omega \mathcal{K}(|x-y|)F(t,\tau)\phi(y,\tau)d\,y d\,\tau + \int_0^t G(t,\tau)\phi(x,\tau)d\,\tau = f(x,t) \tag{1}$$

under the dynamic condition

$$\int_\Omega \phi(x,t)d\,x = p(t) \tag{2}$$

and with initial condition $f(x,0)=0$. Here, $\Omega$ is a non-empty bounded subset of $\mathbb{R}$ and $t \in [0,T]$. The contact problem of a rigid surface $(G,\nu)$ having an elastic material, the integral equation (1) under (2) is investigated from where $G$ is the displacement magnitude and $\nu$ is Poisson's coefficient. If a stamp of length two units where its surface is describing by $f_\star(x)$ , is impressed into an elastic layer surface of a strip by a variable force $p(t)$ , whose eccentricity of application $e(t)$ , that cases rigid displacement $\gamma(t)$. Therefore we define the free term of (1) as

$$f(x,t) = \pi\theta\Big(\delta(t) - f_\star(x)\Big) \quad \theta = \frac{G}{2(1-\nu)}, \quad 0 \leqslant t \leqslant T < 1,\, 0 \leqslant \nu \leqslant 1$$

In (1) the function $F(t,\tau)$ of time represents the resistance force of the lower material, while $G(t,\tau)$ is called the supplied external force in the contact domain of the upper and lower surfaces. In order to guarantee the existence of a unique solution of equation (1), under the condition (2), we assume the following:



- The kernel $\mathcal{K}(|x-y|)$ satisfies the discontinuity condition:

$$\int_{\Omega^2} \mathcal{K}(|x-y|)^2 d\,x d\,y < \infty (i.e \exists)$$

- For all values of $t, \tau \in [0,T]$, the two continuous function of time $F(.,\tau)$ and $G(.,\tau)$ satisfy the following condition:

$$|F(t,\tau)| < B, |G(t,\tau)| < C$$

- The known function $f \in L^2(\Omega) \times C([0,T])$, in this space we define the norm as:

$$||f||_{L^2(\Omega)\times C([0,T])} = \sup_{t\in[0,T]} \int_0^t ||f(.,t)||_{L^2(\Omega)}$$

- The unknown function $\phi$ satisfies Lipschitz condition with respect to the first argument and Hölder condition for the second argument.

In this paper, we use numerical technique based on Trapezoidal rule, to reduce the Volterra-integral Equations to a linear system of Fredhom Integal equations which will be solved using technique of Toeplitz matrices method. The paper is organized as follows. In section 2, we transform the Volterra-fredholm Integral equations to a system of Fredholm integral equations of the second kind. In section, 4, we present our contribution to solve the system of Fredholm Integral equation. In the remainder of the paper, we give a practical example to certify the validity of the proposed technique.

## 2 System of Fredholm integral equations

Many definite integrals cannot be computed in closed form, and must be approximated numerically. In this way, we will use quadrature method to approximate the integral according to $\tau$ and get a system of Fredholm integral equation.



First, if $t = 0$ the Volterra-Fredholm integral equations is verified since $f(x, 0) = 0$. For the sake of convenience, we often assume that the interval $[0, T]$ has been decompose into $m$ pieces of equal length, for some positive integer $m$. The length of each piece is the $\tilde{h} = \frac{T}{m}$, so $t_i = i\,\tilde{h}, i = 0, \ldots, m$. For a given $t = t_j$, we divide the interval of integration $(0; t_j)$ into $j$ equal subintervals, $\delta\tau = \tilde{h} = \frac{t_j - 0}{j}$. Let $\tau_0 = 0, t_0 = \tau_0, t_j = \tau_j = t, \tau_j = j\delta\tau, t_j = \tau_j$. Here,

$$\delta\tau = \frac{\tau_j - 0}{j} = \frac{t - 0}{m}, \tau_j \leqslant t, j \geqslant 1, t = t_m = \tau_m$$

In all our approximation, the error assumed negligible, this help us to get a system of Fredholm Integral equations.

Now, Putting $t = t_j$ in (1) we get:

$$\sum_{i=0}^{j} \nu_i G(t_j, t_i)\phi(x, t_i) + \sum_{i=0}^{j} u_i F(t_j, t_i) \int_{\Omega} \mathcal{K}(|x - y|)\phi(y, t_i)dy = f(x, t_j), j = 0, \ldots, m$$

Here, $t_i$ are the quadrature points and $\nu_i$ and $u_i$ are the quadrature weights. In our study, we will use Trapezoidal integration as quadrature method but others quadrature methods can be applied. In the next we will use these notations:

$$G(t_j,t_i)=G_{j,i},\quad F(t_j,t_i)=F_{j,i},\phi(x,t_i)=\phi_i(x),\quad f(x,t_i)=f_i(x),i,j=0,\dots,m$$

The system of Fredholm Integral equations can be written as:

$$\begin{aligned}\sum_{i=0}^{j}\nu_i G_{j,i}\phi_i(x)+\sum_{i=0}^{j}u_i F_{j,i}\int_\Omega \mathcal{K}(|x-y|)\phi_i(y)dy &= f_j(x),j=0,\dots,m && (3)\\ \int_\Omega \phi_j(x)dx &= p_j,\ j=0,\dots,m && (4)\end{aligned}$$

This lead to: for all $j=1\dots m$, we have the system of Fredholm Integral equation:

$$\begin{aligned}\frac{\tilde{h}}{2}G_{j,j}\phi_j(x)+\frac{\tilde{h}}{2}F_{j,j}\int_\Omega \mathcal{K}(|x-y|)\phi_j(y)dy &= g_j(x) && (5)\\ g_j(x)=f_j(x)-\tilde{h}\sum_{i=0}^{j-1}{}' G_{j,i}\phi_i(x) &- \tilde{h}\sum_{i=0}^{j-1}{}' F_{j,i}\int_\Omega \mathcal{K}(|x-y|)\phi_i(y)dy && (6)\end{aligned}$$



where the prime indicates that the first term to be halved.

A simple modification of (5) leads to:

$$\left\{\begin{aligned}\text{for } j=1,\qquad \frac{\tilde{h}}{2}G_{1,1}\phi_1(x)+\frac{\tilde{h}}{2}F_{1,1}\int_\Omega \mathcal{K}(|x-y|)\phi_1(y)dy &= g_1(x)\\ \text{with}\qquad f_1(x)-\frac{\tilde{h}}{2}G_{1,0}\phi_0(x)-\frac{\tilde{h}}{2}F_{1,0}\int_\Omega \mathcal{K}(|x-y|)\phi_0(y)dy &= g_1(x)\\ \text{for } j=2,\dots,m,\qquad \frac{\tilde{h}}{2}G_{j,j}\phi_j(x)+\frac{\tilde{h}}{2}F_{j,j}\int_\Omega \mathcal{K}(|x-y|)\phi_j(y)dy &= g_j(x)\\ f_j(x)-\tilde{h}\sum_{i=0}^{j-1}{}' G_{j,i}\phi_i(x)-\tilde{h}\frac{F_{j,0}}{2}\int_\Omega \mathcal{K}(|x-y|)\phi_0(y)dy-\tilde{h}\sum_{i=1}^{j-1}F_{j,i}\left(\frac{2}{\tilde{h}F_{i,i}}g_i(x)-\frac{G_{i,i}}{F_{i,i}}\phi_i(x)\right) &= g_j(x)\end{aligned}\right.$$

# 3 Fredholm integral equation

Our Volterra-Fredholm Integral equation is transformed into a system of Fredholm integral equation. Here, we present our strategy to solve the system and for simplicity we will consider the Fredholm Integral equation of the second kind:

$$\phi_j(x) = f_j(x) + \lambda \int_{-a}^{a} \mathcal{K}(|x-y|)\,\phi_j(y)\,dy \tag{7}$$

The interval $[-a, a]$ is divided into $2\,N$ subintervals with constant stepsize $h = \frac{a}{N}$.

Now, we will present three methods to approximate this integral, and we will compute the numerical error given from each method.

$$\begin{aligned}\int_{-a}^{a} \mathcal{K}(|x-y|)\,\phi_j(y)\,dy &= \sum_{n=-N}^{N-d} \left( \int_{nh}^{(n+d)h} k(|x-y|)\,\phi_j(y)\,dy \right) \\ &= \sum_{n=-N}^{N-d} \sum_{i=1}^{d+1} \beta_i^n(x)\phi_j\left((n+i-1)h\right) + \text{Error}\end{aligned} \tag{8}$$



where $\beta_i^n$ are $d+1$ functions will be determined and $Error$ is the estimate error which depends on $x$ and on the way that the coefficients $\beta_n^i$ are chosen. The coefficients are specified such that the local error is $O(h^{d+2})$ and therefore the global error is $O(h^{d+1})$. In order to determine the function $\beta_i^n$ we put in (9) $\phi_i(x) = x^i, i = 1, \ldots, d$, this yields to a Vandermonde linear system in terms of of the functions $\beta_i^n$. For choosing the value of $\phi$ the Error term must be vanish. Here, we limit our presentation to the case where $d \in \{1, 2, 3\}$. The same method can be applied to $d \geqslant 4$.

# 4 Approximation

In this section, we present different choice of the parameter $d$ and the Toeplitz matrix involved in our computation. Our contribution is to develop a higher order approximation (increase the parameter $d$) in order the decrease the error. For this reason, we will consider the known case $d = 1$ and other news cases $d = 2$, and $d = 3$.

### 4.1 First case: $d = 1$

In this case, we have:

$$\int_{nh}^{(n+1)h} K\left(|x-y|\right)\phi_j\left(y\right)dy = \sum_{i=1}^{2}\beta_i^n(x)\phi_j(n+i-1)h \tag{9}$$

In order to determine the function $\beta_i^n, i = 1, 2$. We put in (9) $\phi(x) = 1$ and $\phi(x) = x$, we get the following linear system:

$$\begin{aligned} \int_{nh}^{(n+1)h} \mathcal{K}\left(|x-y|\right)dy &= \beta_1 + \beta_2 \\ \int_{nh}^{(n+1)h} \mathcal{K}\left(|x-y|\right)ydy &= \beta_1 nh + \beta_2\left(n+1\right)h \end{aligned}$$

A simple computation gives:

$$\begin{aligned} \beta_1^n(x) &= (n+1)\int_{nh}^{(n+1)h} \mathcal{K}\left(|x-y|\right)dy - \frac{1}{h}\int_{nh}^{(n+1)h} \mathcal{K}\left(|x-y|\right)ydy \\ \beta_2^n(x) &= -n\int_{nh}^{(n+1)h} \mathcal{K}\left(|x-y|\right)dy + \frac{1}{h}\int_{nh}^{(n+1)h} \mathcal{K}\left(|x-y|\right)ydy \end{aligned}$$

After forgetting the error term (8), we have:

$$\begin{aligned} \int_{-a}^{a} \mathcal{K}\left(|x-y|\right)\phi_j\left(y\right)dy &= \sum_{n=-N}^{N-1}\beta_1^n(x)\phi_j(nh) + \beta_2^n(x)\phi_j((n+1)h) \\ &= \sum_{n=-N}^{N} D_n^2(x)\phi_j(nh) \end{aligned} \tag{10}$$

Where

$$D_n^2(x) = \begin{cases} \beta_1^{-N}(x) & \text{if } n = -N \\ \beta_1^n(x) + \beta_2^{n-1}(x) & \text{if } -N+1 \leqslant n \leqslant N-1 \\ \beta_2^{N-1}(x) & \text{if } n = N \end{cases}$$

Putting $x = m\,h, \forall m = -N, \ldots, N$ in (10), then (7) is a system of $2N+1$ linear equations with $2N+1$ unknowns and can be written as

$$\phi_j(m\,h) - \lambda\sum_{n=-N}^{N} D_2^n(m\,h)\phi_j(nh) = f(m\,h), \forall m \in [\![-N, N]\!] \tag{11}$$

Here, for all $m = -N, \ldots, N$, $\phi(m\,h)$, are the unknowns, $D_2^n(m\,h)$ are the coefficients of the system, and $f(m\,h)$ are the constants term. We use the following notations: for $i = 1, 2$, and for $-N \leqslant m, n \leqslant N, \beta^i_{m,n} = \beta^n_i(m\,h)$.

We define the sequence $\phi_j(m\,h) = \phi_m, f_j(m\,h) = f_{j,m}$ and $D^2_n(m\,h) = D^2_{m,n}$, then the linear system can be written as:

$$\phi_{j,m} - \lambda \sum_{n=-N}^{N} D^2_{m,n}\phi_{j,n} = f_m, \forall m \in [\![-N, N]\!] \tag{12}$$

The $2N + 1$ equations is equivalent to a matrix equation of the form:

$$(\mathbf{Id} - \lambda \mathbf{A})\,\phi_{\mathbf{j}} = \mathbf{f}$$

where $\mathbf{Id}$ is the identity matrix in $\mathcal{M}_{2N+1}(\mathbb{R})$ and $\mathbf{A} \in \mathcal{M}_{2N+1}(\mathbb{R})$, $\phi$ is a column vector with $2\,N + 1$ entries, and $\mathbf{f}$ is a column vector with $2\,N + 1$ entries.



$$\mathbf{A} = \begin{bmatrix} D^2_{-N,-N} & D^2_{-N,-N+1} & \cdots & D^2_{-N,N} \\ D^2_{-N+1,-N} & D^2_{-N+1,-N+1} & \cdots & D^2_{-N+1,N,} \\ \vdots & \vdots & \ddots & \vdots \\ D^2_{N,-N} & D^2_{N,-N+1} & \cdots & D^2_{N,N} \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_{j,-N} \\ \phi_{j,-N+1} \\ \vdots \\ \phi_{j,N} \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_{j,-N} \\ f_{j,-N+1} \\ \vdots \\ f_{j,N} \end{bmatrix}$$

The Matrix $\mathbf{A}$ can be rewritten as:

$$\mathbf{A} = \begin{bmatrix} \beta^1_{-N,-N} & \beta^1_{-N,-N+1} + \beta^2_{-N,-N} & \cdots & \beta^2_{-N,N-1} \\ \beta^1_{-N+1,-N} & \beta^1_{-N+1,-N+1} + \beta^2_{-N+1,-N} & \cdots & \beta^2_{-N+1,N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta^1_{N,-N} & \beta^1_{N,-N+1} + \beta^2_{N,-N} & \cdots & \beta^2_{N,N-1} \end{bmatrix},$$

The matrix $\mathbf{A}$ can be written using a Toeplitz matrix. $\mathbf{A} = \mathbf{T} - \mathbf{E}$, where Putting $x = m\,h$, then the matrix $\mathbf{A}$ can be written using a Toeplitz matrix: $\mathbf{A} = \mathbf{T} - \mathbf{E}$, where

$$T = (T_{mn}), T_{mn} = \beta^1_{m,n} + \beta^2_{m\,n-1} \text{ if } -N \leqslant n, m \leqslant N$$

is a Toeplitz matrix of order $2N + 1$ and

$$E = (E_{mn}), E_{mn} = \begin{cases} \beta^2_{m,-N-1} & \text{if } n = -N,\ -N \leqslant m \leqslant N \\ 0 & \text{if } -N+1 \leqslant n \leqslant N-1,\ -N \leqslant m \leqslant N \\ \beta^1_{m,N} & \text{if } n = N,\ -N \leqslant m \leqslant N \end{cases}$$

represents a matrix of order $2N+1$ whose elements are zeros except the first and last columns.

However, the solution of the system (12) can be obtained in the form

$$\phi_j = (\mathbf{Id} - \lambda(\mathbf{T} - \mathbf{E}))^{-1}\mathbf{f}, \text{ and } |\mathbf{Id} - \lambda\mathbf{A}| \neq 0$$

To clarify the previous idea, let us consider the case $N = 2$, the matrix $\mathbf{A} \in \mathcal{M}_5(\mathbb{R})$ can be written as:

$$\mathbf{A} = \begin{bmatrix} \beta^1_{-2,-2} & \beta^1_{-2,-1} + \beta^2_{-2,-2} & \beta^1_{-2,0} + \beta^2_{-2,-1} & \beta^1_{-2,1} + \beta^2_{-2,0} & \beta^2_{-2,1} \\ \beta^1_{-1,-2} & \beta^1_{-1,-1} + \beta^2_{-1,-2} & \beta^1_{-1,0} + \beta^2_{-1,-1} & \beta^1_{-1,1} + \beta^2_{-1,0} & \beta^2_{-1,1} \\ \beta^1_{0,-2} & \beta^1_{0,-1} + \beta^2_{0,-2} & \beta^1_{0,0} + \beta^2_{0,-1} & \beta^1_{0,1} + \beta^2_{0,0} & \beta^2_{0,1} \\ \beta^1_{1,-2} & \beta^1_{1,-1} + \beta^2_{1,-2} & \beta^1_{1,0} + \beta^2_{1,-1} & \beta^1_{1,1} + \beta^2_{1,0} & \beta^2_{1,1} \\ \beta^1_{2,-2} & \beta^1_{2,-1} + \beta^2_{2,-2} & \beta^1_{2,0} + \beta^2_{2,-1} & \beta^1_{2,1} + \beta^2_{2,0} & \beta^2_{2,1} \end{bmatrix}$$

in this case the Toeplitz matrix $T$ is given by:

$$\mathbf{T} = \begin{bmatrix} \beta^1_{-2,-2} + \beta^2_{-2,-3} & \beta^1_{-2,-1} + \beta^2_{-2,-2} & \beta^1_{-2,0} + \beta^2_{-2,-1} & \beta^1_{-2,1} + \beta^2_{-2,0} & \beta^1_{-2,2} + \beta^2_{-2,1} \\ \beta^1_{-1,-2} + \beta^2_{-1,-3} & \beta^1_{-1,-1} + \beta^2_{-1,-2} & \beta^1_{-1,0} + \beta^2_{-1,-1} & \beta^1_{-1,1} + \beta^2_{-1,0} & \beta^1_{-1,2} + \beta^2_{-1,1} \\ \beta^1_{0,-2} + \beta^2_{0,-3} & \beta^1_{0,-1} + \beta^2_{0,-2} & \beta^1_{0,0} + \beta^2_{0,-1} & \beta^1_{0,1} + \beta^2_{0,0} & \beta^1_{0,2} + \beta^2_{0,1} \\ \beta^1_{1,-2} + \beta^2_{1,-3} & \beta^1_{1,-1} + \beta^2_{1,-2} & \beta^1_{1,0} + \beta^2_{1,-1} & \beta^1_{1,1} + \beta^2_{1,0} & \beta^1_{1,2} + \beta^2_{1,1} \\ \beta^1_{2,-2} + \beta^2_{2,-3} & \beta^1_{2,-1} + \beta^2_{2,-2} & \beta^1_{2,0} + \beta^2_{2,-1} & \beta^1_{2,1} + \beta^2_{2,0} & \beta^1_{2,2} + \beta^2_{2,1} \end{bmatrix}$$

and the matrix $E$ is nothing but:

$$\mathbf{E} = \begin{bmatrix} \beta^2_{-2,-3} & 0 & 0 & 0 & \beta^1_{-2,2} \\ \beta^2_{-1,-3} & 0 & 0 & 0 & \beta^1_{-1,2} \\ \beta^2_{0,-3} & 0 & 0 & 0 & \beta^1_{0,2} \\ \beta^2_{1,-3} & 0 & 0 & 0 & \beta^1_{1,2} \\ \beta^2_{2,-3} & 0 & 0 & 0 & \beta^1_{2,2} \end{bmatrix}$$

## 4.2 Case $d = 2$

The case Case $d = 2$ is considered by M. A. DARWISH in [11], the author use the same technique presented by some others authors (in the case $d = 1$), but unfortunately there is an error occurred in the computation. In fact, the author does not introduce the right matrix to be used letter in the numerical example, but he use the transposed matrix, exactly, he defined the matrix $a'_{n,m}$ as: *the elements of the second matrix are zeros except the elements of the first two rows and last two rows* this is not true. Here, we present the right form of the Toeplitz matrices as well as the other matrix used in the computation.

The same method presented in the case $d = 1$ is applied and after neglecting the error term we can write:

$$\int_{nh}^{(n+2)h} \mathcal{K}\left(|x-y|\right)\phi\left(y\right)dy = \sum_{i=1}^{3}\beta_i^n(x)\phi(n+i-1)h \tag{13}$$

In order to determine the function $\beta_i^n, i = 1, 2, 3$. We put in (13) $\phi(x) = x^i, i = 0, 1, 2$, we get the following Vandermonde system:

$$\begin{pmatrix} 1 & 1 & 1 \\ nh & (n+1)\,h & (n+2)\,h \\ n^2h^2 & (n+1)^2\,h^2 & (n+2)^2\,h^2 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} I_0^n(x) \\ I_1^n(x) \\ I_2^n(x) \end{pmatrix}$$

with

$$I_i^n(x) = \int_{nh}^{(n+2)h} y^i\mathcal{K}\left(|x-y|\right)dy, i = 0, 1, 2$$

whose solution is:

$$\beta_1 = \frac{(n+1)\,(n+2)}{2}I_0^n(x) - \frac{2n+3}{2h}\,I_1^n(x) + \frac{1}{2h^2}\,I_2^n(x) \tag{14}$$

$$\beta_2 = -n\,(n+2)\,I_0^n(x) + \frac{2\,(n+1)}{h}I_1^n(x) - \frac{1}{h^2}\,I_2^n(x) \tag{15}$$

$$\beta_3 = \frac{n(n+1)}{2}I_0^n(x) + \frac{(2\,n+1)}{2h}I_1^n(x) + \frac{1}{2h^2}I_2^n(x) \tag{16}$$

Then,

$$\begin{aligned}\int_{-a}^{a}\mathcal{K}\left(|x-y|\right)\phi\left(y\right)dy &= \sum_{n=-N}^{N-2}\beta_1^n(x)\phi(nh) + \beta_2^n(x)\phi((n+1)h) + \beta_3^n(x)\phi((n+2)h) \\ &= \sum_{n=-N}^{N} D_n^3(x)\phi(nh)\end{aligned}$$

Where

$$D_n^3(x) = \begin{cases} \beta_1^{-N}(x) & \text{if } n = -N \\ \beta_1^{-N+1}(x) + \beta_2^{-N}(x) & \text{if } n = -N+1 \\ \beta_1^n(x) + \beta_2^{n-1}(x) + \beta_3^{n-2}(x) & \text{if } -N+2 \leqslant n \leqslant N-2 \\ \beta_2^{N-2}(x) + \beta_3^{N-3}(x) & \text{if } n = N-1 \\ \beta_3^{N-2}(x) & \text{if } n = N \end{cases}$$

Following the method presented with $d = 1$, we have:

$$\phi_m - \lambda \sum_{n=-N}^{N} D^3_{m,n}\phi_n = f_m, \forall m \in [\![-N, N]\!] \tag{17}$$

The $2N + 1$ equations is equivalent to a matrix equation of the form:

$$(\mathbf{Id} - \lambda \mathbf{A})\,\phi = \mathbf{f}$$

where $\mathbf{Id}$ is the identity matrix in $\mathcal{M}_{2N+1}(\mathbb{R})$ and $\mathbf{A} \in \mathcal{M}_{2N+1}(\mathbb{R})$, $\phi$ is a column vector with $2\,N + 1$ entries, and $\mathbf{f}$ is a column vector with $2\,N + 1$ entries.

$$\mathbf{A} = \begin{bmatrix} D^3_{-N,-N} & D^3_{-N,-N+1} & \cdots & D^3_{-N,N} \\ D^3_{-N+1,-N} & D^3_{-N+1,-N+1} & \cdots & D^3_{-N+1,N} \\ \vdots & \vdots & \ddots & \vdots \\ D^3_{N,-N} & D^3_{N,-N+1} & \cdots & D^3_{N,N} \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_{-N} \\ \phi_{-N+1} \\ \vdots \\ \phi_N \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_{-N} \\ f_{-N+1} \\ \vdots \\ f_N \end{bmatrix}$$

The matrix $\mathbf{A}$ can be written using a Toeplitz matrix,i.e. $\mathbf{A} = \mathbf{T} - \mathbf{E}$, where

$$\mathbf{T} = (T_{mn}), \mathbf{T}_{mn} = \beta^1_{m,n} + \beta^2_{m,n-1} + \beta^3_{m,n-2} \text{ if } -N \leqslant n, m \leqslant N$$

is a Toeplitz matrix of order $2N + 1$ and

$$\mathbf{E} = (\mathbf{E}_{mn}), E_{mn} = \begin{cases} \beta^2_{m,-N-1} + \beta^3_{m,-N-2} & \text{if } n = -N,\ -N \leqslant m \leqslant N \\ \beta^3_{m,-N-1} & \text{if } n = -N+1,\ -N \leqslant m \leqslant N \\ 0 & \text{if } -N+1 \leqslant n \leqslant N-1,\ -N \leqslant m \leqslant N \\ \beta^1_{m,N-1} & \text{if } n = N-1,\ -N \leqslant m \leqslant N \\ \beta^1_{m,N} + \beta^2_{m,N-1} & \text{if } n = N,\ -N \leqslant m \leqslant N \end{cases}$$

$\mathbf{E}$ has the form

$$\mathbf{E} = \begin{bmatrix} \beta^2_{-N,-N-1} + \beta^3_{-N,-N-2} & \beta^3_{-N,-N-1} & 0 & \ldots & 0 & \beta^1_{-N,N-1} & \beta^1_{-N,N} + \beta^2_{-N,N-1} \\ \beta^2_{-N+1,-N-1} + \beta^3_{-N+1,-N-2} & \beta^3_{-N-1,-N-1} & 0 & \ldots & 0 & \beta^1_{-N+1,N-1} & \beta^1_{-N+1,N} + \beta^2_{-N+1,N-1} \\ \vdots & \vdots & 0 & \ldots & 0 & \vdots & \vdots \\ \beta^2_{N,-N-1} + \beta^3_{N,-N-2} & \beta^3_{N,-N-1} & 0 & \ldots & 0 & \beta^1_{N,N-1} & \beta^1_{N,N} + \beta^2_{N,N-1} \end{bmatrix}$$

represents a matrix of order $2N + 1$ and the elements of the matrix $\mathbf{E}$ are zeros except the elements of the first two columns and last two columns.

Like the case $d = 1$, the approximate solution can be obtained in the form

$$\phi = (\mathbf{Id} - \lambda(\mathbf{T} - \mathbf{E}))^{-1}\mathbf{f}, \text{ and } |\mathbf{Id} - \lambda\mathbf{A}| \neq 0$$

### 4.3 Case $d = 3$

Similar to the previous cases, we have the following Vandermonde system:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ nh & (n+1)\,h & (n+2)\,h & (n+3)\,h \\ n^2h^2 & (n+1)^2\,h^2 & (n+2)^2\,h^2 & (n+2)^2\,h^2 \\ n^2h^3 & (n+1)^3\,h^3 & (n+2)^3\,h^3 & (n+2)^3\,h^3 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} I_0^n(x) \\ I_1^n(x) \\ I_2^n(x) \\ I_3^n(x) \end{pmatrix}$$

whose solution is:

$$\begin{aligned}
\beta_1 &= \left(1 + \frac{11}{6}n + n^2 + \frac{1}{6}n^3\right) I_0^n(x) - \frac{11 + 12n + 3n^2}{6h}\, I_1^n(x) + \frac{n+2}{2h^2}\, I_2^n(x) - \frac{1}{6h^3}\, I_3^n(x) \\
\beta_2 &= \frac{-n\,(6 + 5n + n^2)}{2} I_0^n(x) + \frac{(6 + 10n + 3n^2)}{h} I_1^n(x) - \frac{5 + 3n}{2h^2}\, I_2^n(x) + \frac{1}{2h^3}\, I_3^n(x) \\
\beta_3 &= \frac{n(n^2 + 4n + 3)}{2} I_0^n(x) - \frac{(3n^2 + 8n + 3)}{2h} I_1^n(x) + \frac{3n + 4}{2h^2} I_2^n(x) - \frac{1}{2h^3}\, I_3^n(x) \\
\beta_4 &= \frac{-n(n^2 + 3n + 2)}{6} I_0^n(x) + \frac{(3n^2 + 6n + 2)}{6h} I_1^n(x) - \frac{n+1}{2h^2} I_2^n(x) + \frac{1}{6h^3}\, I_3^n(x)
\end{aligned}$$



Then,

$$\begin{aligned}
\int_{-a}^{a} K\left(|x - y|\right) \phi\left(y\right) dy &= \sum_{n=-N}^{N-3} \sum_{i=1}^{4} \beta_i^n(x)\phi((n + i - 1)h) \\
&= \sum_{n=-N}^{N} D_n^4(x)\phi(nh)
\end{aligned}$$

Where

$$D_n^4(x) = \begin{cases} \beta_1^{-N}(x) & \text{if } n = -N \\ \beta_1^{-N+1}(x) + \beta_2^{-N}(x) & \text{if } n = -N+1 \\ \beta_1^{-N+2}(x) + \beta_2^{-N+1}(x) + \beta_3^{-N}(x) & \text{if } n = -N+2 \\ \beta_1^{n}(x) + \beta_2^{n-1}(x) + \beta_3^{n-2}(x) + \beta_4^{n-3}(x) & \text{if } -N+3 \leqslant n \leqslant N-3 \\ \beta_2^{N-3}(x) + \beta_3^{N-4}(x) + \beta_4^{N-5}(x) & \text{if } n = N-2 \\ \beta_3^{N-3}(x) + \beta_4^{N-4}(x) & \text{if } n = N-1 \\ \beta_4^{N-3}(x) & \text{if } n = N \end{cases}$$

Following the method presented with $d=1$ and $d=2$ we have:

$$\phi_m - \lambda \sum_{n=-N}^{N} D^4_{m,n}\phi_n = f_m, \forall m \in [\![-N, N]\!] \tag{18}$$

The $2N+1$ equations is equivalent to a matrix equation of the form:

$$(\mathbf{Id} - \lambda \mathbf{A})\,\phi = \mathbf{f}$$

where $\mathbf{Id}$ is the identity matrix in $\mathcal{M}_{2N+1}(\mathbb{R})$ and $\mathbf{A} \in \mathcal{M}_{2N+1}(\mathbb{R})$, $\phi$ is a column vector with $2\,N+1$ entries, and $\mathbf{f}$ is a column vector with $2\,N+1$ entries.

$$\mathbf{A} = \begin{bmatrix} D^4_{-N,-N} & D^4_{-N,-N+1} & \cdots & D^4_{-N,N} \\ D^4_{-N+1,-N} & D^4_{-N+1,-N+1} & \cdots & D^4_{-N+1,N} \\ \vdots & \vdots & \ddots & \vdots \\ D^4_{N,-N} & D^4_{N,-N+1} & \cdots & D^4_{N,N} \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_{-N} \\ \phi_{-N+1} \\ \vdots \\ \phi_N \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} f_{-N} \\ f_{-N+1} \\ \vdots \\ f_N \end{bmatrix}$$

The matrix $\mathbf{A}$ can be written using a Toeplitz matrix,i.e. $\mathbf{A} = \mathbf{T} - \mathbf{E}$, where

$$\mathbf{T} = (T_{mn}), \mathbf{T}_{mn} = \beta^1_{m,n} + \beta^2_{m,n-1} + \beta^3_{m,n-2} + \beta^4_{m,n-3} \text{ if } -N \leqslant n, m \leqslant N$$

is a Toeplitz matrix of order $2N+1$ and

$$\mathbf{E} = (\mathbf{E}_{mn}), E_{mn} = \begin{cases} \beta^2_{m,-N-1} + \beta^3_{m,-N-2} + \beta^4_{m,-N-3} & \text{if } n=-N,\ -N \leqslant m \leqslant N \\ \beta^3_{m,-N-2} + \beta^4_{m,-N-3} & \text{if } n=-N,\ -N \leqslant m \leqslant N \\ \beta^4_{m,-N-1} & \text{if } n=-N+1,\ -N \leqslant m \leqslant N \\ 0 & \text{if } -N+1 \leqslant n \leqslant N-1,\ -N \leqslant m \leqslant N \\ \beta^1_{m,N-1} & \text{if } n=N-1,\ -N \leqslant m \leqslant N \\ \beta^1_{m,N} + \beta^2_{m,N-1} & \text{if } n=N,\ -N \leqslant m \leqslant N \\ \beta^1_{m,N} + \beta^2_{m,N-1} + \beta^3_{m,N-1} & \text{if } n=N,\ -N \leqslant m \leqslant N \end{cases}$$

represents a matrix of order $2N+1$ and the elements of the matrix $\mathbf{E}$ are zeros except the elements of the first three columns and last three columns.

Like the case $d=1,2$, the approximate solution can be obtained in the form

$$\phi = (\mathbf{Id} - \lambda(\mathbf{T} - \mathbf{E}))^{-1}\mathbf{f}, \text{ and } |\mathbf{Id} - \lambda \mathbf{A}| \neq 0$$

# 5 Numerical results

Here, we present some numerical examples for different value of $N$ and we make some comparison with other methods used for the same problem. First, let us say, that there is many authors present in them works the error versus $x$ with fixed value of $N$. In our study we say it's not interesting to do that, the plotting of error versus $N$ in any fixed point is more interesting. They think that the error is decreasing function according to $x$ (Which has no numerical meaning), and this idea is not generally true, there is no relationship between the error at each point. But, the principal goal is to show that the error is a decreasing function according to the step-size $h$. The Toeplitz matrix method used to solve the Volterra/Fredholm Integral equation is a good idea but they write in a bad way the Toeplitz matrix arises from the linear system, see [11] ( case $d = 2$). Furthermore, they compute the eigenvectors and eigenvalues of the Topelitz matrix which has no relation of the aim of the paper. This paper propose an improvement of some papers and give a more accurate solution of the Volterra/Fredholm integral equation. We will present the local error at each point and plot the error versus $N$ for each case $d = 1, 2, 3$. To achieve the validity, the accuracy and support our theoretical discussion of the proposed method, we give some computational results. The computations, associated with the example, are performed by MATLAB7. Carleman Kernel will be proposed as application. Moreover, we can give more complicate example, and using the software Maple, to make the computation.

In the case of Carleman Kernel, i.e. $\mathcal{K}(|x-y|) = |x-y|^{-\alpha}$, $T = 1$ and $\alpha \in ]0,1[$ .

**Example 1** *Here, we consider the simple case where $F = G = 1$ and the exact solution is $\phi(x,t) = t^2 + x^2$ associated to*

$$f(x,t) = \frac{t^3}{3}\int_{-1}^{1} |x-y|^{-\alpha}dy \, + \, t\int_{-1}^{1} y^2|x-y|^{-\alpha}dy$$

*With*

$$\begin{aligned}
\int_{-1}^{1} |x-y|^{-\alpha} dy &= \tfrac{1}{1-\alpha}\Big((1-x)^{1-\alpha} + (1+x)^{1-\alpha}\Big) \\
\int_{-1}^{1} y^2 |x-y|^{-\alpha} dy &= \tfrac{2}{3-\alpha} x \eta_1(x) + \Big(\tfrac{1}{3} - \tfrac{2\alpha}{3(3-\alpha)}\Big)\eta_0(x) \\
\eta_0 &= \tfrac{1}{1-\alpha}\Big((1-x)^{1-\alpha} + (1+x)^{1-\alpha}\Big) \\
\eta_1 &= x\,\eta_0 + \tfrac{1}{2-\alpha}\Big((1-x)^{2-\alpha} + (1+x)^{2-\alpha}\Big)
\end{aligned}$$

*In our computation we will take* $\alpha = 0.5$.

*In the case* $d = 1$, *Table 1 gives the error at the fixed point* $(x = 1, t = T)$, *the error is deceasing according to* $N$.

| N | 10 | 20 | 30 | 40 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|---|
| Error | 2.423E-2 | 2.117E-2 | 4.982E-3 | 3.214E-3 | 7.581E-4 | 5.981E-4 | 2.982E-5 |

Table 1: Case $d = 1$. Comparison of the $Error = |\phi_N - \phi_{exact}|$ with respect to the number $N$

*In the case* $d = 2$, *Table 1 gives the error at the fixed point* $(x = 1, t = T)$, *we see that the error is a deceasing according to* $N$.

| N | 10 | 20 | 30 | 40 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|---|
| Error | 2.123E-2 | 2.026E-3 | 1.125E-4 | 4.447E-5 | 1.216E-5 | 3.247E-6 | 1.126E-7 |

Table 2: Case $d = 2$. Comparison of the $Error = |\phi_N - \phi_{exact}|$ with respect the number $N$



*In the case* $d = 3$, *Table 1 gives the error at the fixed point* $(x = 1, t = T)$, *we see that the error is a deceasing according to* $N$.

| N | 10 | 20 | 30 | 40 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|---|
| Error | 1.012E-3 | 1.011E-4 | 2.345E-5 | 8.760E-6 | 1.216E-6 | 8.346E-8 | 1.307E-8 |

Table 3: Case $d = 3$. Comparison of the $Error = |\phi_N - \phi_{exact}|$ with respect to the number $N$

**Example 2** *Here, we consider* $\alpha = \frac{1}{2}$. *Consider the Volterra-Fredholm Integral equation of second kind with:*

$$F(x,t) = 1,\ G(x,t) = 1,\ f(x,t) = \ln|1+t| \int_{-1}^{1} |x-y|^{-\alpha} dy - t \int_{-1}^{1} y^3\, |x-y|^{-\alpha} dy$$

*where*

$$\begin{aligned}
\int_{-1}^{1} y^3 \left| x-y \right|^{-\alpha} dy &= x^3 + \Big(\omega_1 + \omega_2\Big) \\
\omega_1 &= -\tfrac{1}{7}x_1^7 + \tfrac{3}{5}xx_1^5 - x^2x_1^3 + x^3x_1 \\
\omega_2 &= \tfrac{1}{7}x_2^7 + \tfrac{3}{5}xx_2^5 + x^2x_2^3 + x^3x_2 \\
x_1 &= \sqrt{1+x} \\
x_2 &= \sqrt{1-x}
\end{aligned}$$

*and the exact solution is* $\phi(x,t) = \dfrac{1}{1+t} - x^3$. *The plot of* $\log_{10}\left|e^N\right|$ *with respect to* $N$ *(see Figure 1) shows the improvement brought by* $d = 3$. *The Figure 2 and 3 shows the exact and approximate solutions in both example with* $d = 3$ *and* $N = 50$.

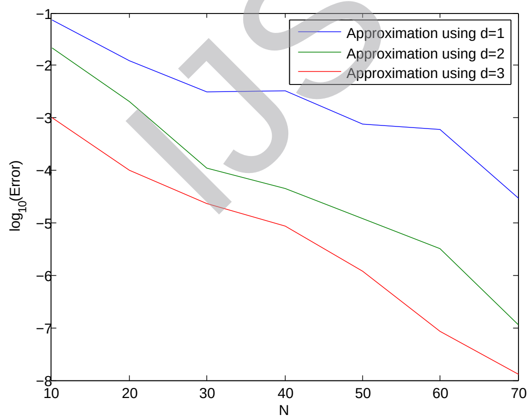



Figure 1: The error versus $N$

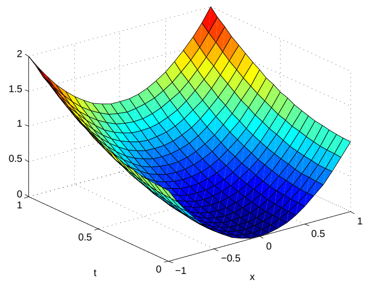

Figure 2: Case $\phi(x) = t^2 + x^2$. Exact and approximate solution with $N = 50$ and $d = 3$.

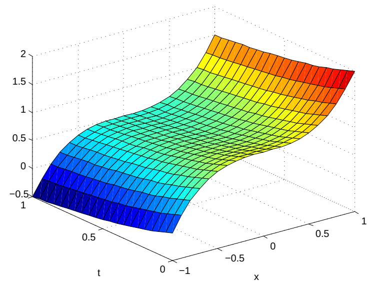

Figure 3: Case $\phi(x) = \frac{1}{1+t} - x^3$. Exact and approximate solution with $N = 50$ and $d = 3$.

# 6 Conclusion



1. Note, the case $d = 3$, never has been considered by other author. It's give an improvement in the error term.

2. As we see, the error for the three cases $d = 1, 2, 3$ is a decreasing function according to $N$. The case $d = 3$ is better than the other ones.

3. The method presented in our paper can be generalized to more general Kernels.

4. Our method can be compared to other method like Chebychev and Legendre approximations. This will be considered in other papers.

# References

[1] R.Grimmer, J. H. Liu, Singular perturbations in viscoelasticity, Rocky Mountain J. Math. 24(1994),61-75 .

[2] W. E. Olmstead and R. A. Handelsman, Diffusion in a semi-infinite region with nonlinear surface dissipation, SIAM Rev. 18(1996),275-291.

[3] M.A.Abdou, Integral equation with Macdonald kernel and its application to a system of contact problem, J.Appl.Math.Comput. 118(2001)83-94.

[4] M.A.Abdou, Spectral relationships for the integral equation with Macdonald kernel and contact problem J.Appl.Math.Comput. 118(2002)93-103.

[5] M.A.Abdou, A.A.Badr, On a method for solving an integral equation in the displacement contact problem, J.Appl.Math.Comput 127(2002)65-78.

[6] M.A.Abdou, Fredholm-Volterra integral equation and generalized potential kernel, J.Appl. Math. Comput. 131(2003) 81-94.

[7] H. Hochstadt, Integral Equations, John Wiley, New York 1973.

[8] R.P. Kanwal, Linear Integral Equations, Academic Press, New York, 1991.

[9] K.E. Atkinson, A Survey of Numerical Methods for the Solution of Fredholm Integral Equations of the Second Kind, Society for Industrial and Applied Mathematics , Philadelphia, PA.1976.

[10] L.M. Delves and J.l. Mohamed, Computational Methods for Integral Equations, Cambridge University Press, Cambridge, 1985.

[11] M. A. DARWISH. Fredholm-Volterra equation with singular kernel. Korean J. Comput. Appl. Math. Vol. 6 (1999), No.1, pp. 163-174.