A NOTE ON COMPLETE K – REGULAR GRAPH AND HAMILTONIAN CYCLE

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ABSTRACT

In 1856, Hamiltonian introduced the Hamiltonian Graph where a Graph which is covered all the vertices without repetition and end with starting vertex. In this paper I would like to prove Every Complete K – Regular graph of order n ≥ 3 such that 2k ≥ n then it is Hamiltonian Cycle.

Key words: Graph, Euler Graph, Hamiltonian graph, Regular Graph, Complete Graph.

1. Introduction:

The origin of graph theory started with the problem of Koinsber bridge, in 1735.

This problem lead to the concept of Eulerian Graph. Euler studied the problem of Koinsberg bridge and constructed a structure to solve the problem called Eulerian graph.

In 1840, A.F Mobius gave the idea of complete graph and bipartite graph and Kuratowski proved that they are planar by means of recreational problems.

The concept of tree, (a connected graph without cycles was implemented by Gustav Kirchhoff in 1845, and he employed graph theoretical ideas in the calculation of currents in electrical networks or circuits.
In 1852, Thomas Gutherie found the famous four color problem.

Then in 1856, Thomas P. Kirkman and William R. Hamilton studied cycles on polyhedra and invented the concept called Hamiltonian graph by studying trips that visited certain sites exactly once.

In 1913, H. Dudeney mentioned a puzzle problem. Even though the four color problem was invented it was solved only after a century by Kenneth Appel and Wolfgang Haken.

This time is considered as the birth of Graph Theory.

Caley studied particular analytical forms from differential calculus to study the trees. This had many implications in theoretical chemistry. This lead to the invention of enumerative graph theory.

Any how the term “Graph” was introduced by Sylvester in 1878 where he drew an analogy between “Quantic invariants” and covariants of algebra and molecular diagrams.

In 1941, Ramsey worked on colorations which lead to the identification of another branch of graph theory called extremal graph theory.

In 1969, the four color problem was solved using computers by Heinrich. The study of asymptotic graph connectivity gave rise to random graph theory.

1.1 Definition: A graph – usually denoted \( G(V,E) \) or \( G = (V,E) \) – consists of set of vertices \( V \) together with a set of edges \( E \). The number of vertices in a graph is usually denoted \( n \) while the number of edges is usually denoted \( m \).

1.2 Definition: Vertices are also known as nodes, points and (in social networks) as actors, agents or players.

1.3 Definition: Edges are also known as lines and (in social networks) as ties or links. An edge \( e = (u,v) \) is defined by the unordered pair of vertices that serve as its end points.

1.4 Example: The graph depicted in Figure 1 has vertex set \( V = \{a,b,c,d,e,f\} \) and edge set \( E = \{(a,b),(b,c),(c,d),(c,e),(d,e),(e,f)\} \).

![Figure 1](image-url)
1.5 Definition: Two vertices $u$ and $v$ are adjacent if there exists an edge $(u,v)$ that connects them.

1.6 Definition: An edge $(u,v)$ is said to be incident upon nodes $u$ and $v$.

1.7 Definition: An edge $e = (u,u)$ that links a vertex to itself is known as a self-loop or reflexive tie.

1.8 Definition: Every graph has associated with it an adjacency matrix, which is a binary $n \times n$ matrix $A$ in which $a_{ij} = 1$ and $a_{ji} = 1$ if vertex $v_i$ is adjacent to vertex $v_j$, and $a_{ij} = 0$ and $a_{ji} = 0$ otherwise. The natural graphical representation of an adjacency matrix is a table, such as shown below.

\[
\begin{array}{cccccc}
  & a & b & c & d & e \\
 a & 0 & 1 & 0 & 0 & 0 \\
b & 1 & 0 & 1 & 0 & 0 \\
c & 0 & 1 & 0 & 1 & 1 \\
d & 0 & 0 & 1 & 0 & 1 \\
e & 0 & 0 & 1 & 1 & 0 \\
f & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

Adjacency matrix for graph in Figure 1.

1.9 Definition: Examining either Figure 1 or given adjacency Matrix, we can see that not every vertex is adjacent to every other. A graph in which all vertices are adjacent to all others is said to be complete.

1.10 Definition: While not every vertex in the graph in Figure 1 is adjacent, one can construct a sequence of adjacent vertices from any vertex to any other. Graphs with this property are called connected.
1.11 **Note:** Reachability. Similarly, any pair of vertices in which one vertex can reach the other via a sequence of adjacent vertices is called *reachable*. If we determine reachability for every pair of vertices, we can construct a reachability matrix $R$ such as depicted in Figure 2. The matrix $R$ can be thought of as the result of applying transitive closure to the adjacency matrix $A$.

![Figure 2](image)

**Figure 2**

1.12 **Definition:** A walk is closed if $v_0 = v_n$.

1.13 **Definition:** A *tree* is a connected graph that contains no cycles. In a tree, every pair of points is connected by a unique path. That is, there is only one way to get from $A$ to $B$.

![Figure 3](image)

**Figure 3:** A labeled tree with 6 vertices and 5 edges

1.14 **Definition:** A *spanning tree* for a graph $G$ is a sub-graph of $G$ which is a tree that includes every vertex of $G$.

1.15 **Definition:** The length of a walk (and therefore a path or trail) is defined as the number of edges it contains. For example, in Figure 3, the path $a,b,c,d,e$ has length 4.
1.16 Definition: The number of vertices adjacent to a given vertex is called the *degree* of the vertex and is denoted \( d(v) \).

1.17 Definition: In the mathematical field of *graph theory*, a *bipartite graph* (or *bigraph*) is a graph whose *vertices* can be divided into two *disjoint sets* \( U \) and \( V \) such that every *edge* connects a vertex in \( U \) to one in \( V \); that is, \( U \) and \( V \) are *independent sets*. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles.

![Figure 4: Example of a bipartite graph.](image)

1.18 Definition: An *Eulerian circuit* in a graph \( G \) is a circuit which includes every vertex and every edge of \( G \). It may pass through a vertex more than once, but because it is a circuit it traverses each edge exactly once. A graph which has an Eulerian circuit is called an *Eulerian graph*. An *Eulerian path* in a graph \( G \) is a walk which passes through every vertex of \( G \) and which traverses each edge of \( G \) exactly once.

1.19 Example: Königsberg bridge problem: The city of Königsberg (now Kaliningrad) had seven bridges on the Pregel River. People were wondering whether it would be possible to take a walk through the city passing exactly once on each bridge. Euler built the representative graph, observed that it had vertices of odd degree, and proved that this made such a walk impossible. Does there exist a walk crossing each of the seven bridges of Königsberg exactly once?

![Figure 5: Königsberg problem](image)

1.20 Definition: *Complete Graph*: A simple graph in which there exists an edge between every pair of vertices is called a complete graph.
1.21 Definition: Regular Graph: A graph is regular if all the vertices of $G$ have the same degree. In particular, if the degree of each vertex is $r$, the $G$ is regular of degree $r$.

1.22 Example:

![Graphs with degree $r = 0$](image1)

$Graphs \ with\ degree\ r = 0$

![Graphs with degree $r = 2$](image2)

$Graphs \ with\ degrees\ r = 2$

![Graphs with degree $r = 3$](image3)

$Graphs \ with\ degrees\ r = 3$

1.23 Definition: A Hamilton circuit is a path that visits every vertex in the graph exactly once and return to the starting vertex. Determining whether such paths or circuits exist is an NP-complete problem. In the diagram below, an example Hamilton Circuit would be
1.24 Example:

![Diagram of a graph with vertices A, B, C, D, E, F, G and edges connecting them.]

Figure 6: Hamilton Circuit would be AEFGCDBA.

1.25 Theorem:

Let ‘G’ be a Complete K-Regular graph of order \( n \geq 3 \) such that \( 2k \geq n \), then ‘G’ is a Hamiltonian.

Proof:

Let ‘G’ be a Complete k-regular graph with \( k \geq n/2 \).

Assume that ‘G’ contains no Hamiltonian cycle.

Let ‘G’ be the graph obtained from G by adding a maximum number of edges creating without a cycle.

i.e., Any edge adding to ‘G’ will create a Hamilton cycle.

Let \( u, v \in V \) be two non-adjacent vertices in ‘G’ and let ‘P’ be the Hamilton path

\[ P = (u = v_1, v_2, \ldots, v_n = v) \]

Define any two sets

\[ S = \{ i : V_{i+1} \in N(u) \} \quad \text{and} \quad T = \{ i : V_i \in N(v) \} \]

Clearly \( S \cap T \neq \emptyset \)

Otherwise \( S \cup T = n \) with \( n \notin S \cup T \)

Let \( m \in S \cap T \)
Then \((u,v_2,\ldots,v_m,v,v_{n-1},\ldots,v_{m+1},v)\) is a Hamiltonian cycle in ‘\(G^1\)’

Which is a contradiction to our assumption

Therefore we cannot extend ‘\(G\)’ to a maximal graph ‘\(G^1\)’

And we may conclude that ‘\(G\)’ contains a Hamiltonian Cycle.

Hence the Theorem.

References:


