International Journal of Computer Application (2250-1797)
Volume 5– No. 4, June- 2015

ON THE SUPER MAGIC DEFICIENCY OF SOME FORESTS AND 2- REGULAR GRAPHS

MALLIKARJUN GHALEPPA AND DANAPPA G AKKA

ABSTRACT. A(p,q) graph G is called $\underline{edge-magic}$ if there exists a bijection $f : V(G) \cup E(G) \to \{1, 2, - - -, p+q\}$ such that for each edge $e = (u, v)$ of G, we have $f(u) + f(v) + f(uv)$ as a constant. Also G is said to be $\underline{super-edge-magic}$ if $V(G) = \{1, 2, - - -, P\}$. Further G is said to be $\underline{Super-magic}$ if $E(G) = \{1, 2, - - -, q\}$. The $\underline{Super-magic-deficiency}$, $\mu_i(G)$ of a graph G is defined to be either the smallest non negative integer n with the property that the graph $G \cup nK_1$ is super magic or $+\infty$ if there exists no such integer.

In this paper, the super magic deficiency of certain forests and 2-regular graphs is established. In addition some edge magic deficiency analogues to the super magic deficiency results on forests are obtained.

1. INTRODUCTION

All graphs considered in this paper are finite and simple. That is to say, they do not contain neither loops nor multiple edges. Also for additional terminology and notation not specified in this paper, we refer the reader to either [7] or [11].

In 1970, Kotzig and Rosa [14] defined a graph $G = (V, E)$ of order p and size q to be edge- magic if there exists a bijiective function $f : V(G) \cup E(G) \to \{1, 2, - - -, p+q\}$ such that for any edge $uv \in E(G)$, we have that $f(u) + f(uv) + f(v)$ is constant. They called such function an edge magic labeling of G. Later on in 1998, Enomoto et al [8] motivated by the previous definition, defined the concept of super edge magic labeling as follows. A Graph $G = (V, E)$ of order p and size q is super edge magic if there exists an edge-magic labeling f of G with the extra property that $f(V(G)) = \{1, 2, - - -p\}$ such

Key words and phrases. Regular graph, Edge magic, Super magic, Super magic deficiency.

AMS Subject classificition: Primary 05B15, 05B20, Secondary 62K05.

labeling f is called a super edge magic labeling of G.
Similarly in 2010, Akka et al [1] motivated by the edge magic definition, defined the concept of super magic labeling as follows . A graph $G = (V, E)$ of order p and size q is super magic if there exists an edge magic labeling f of G with extra property that $f(E) = \{1, 2, - - -, q\}$ such labeling is called a super magic labeling of G.

In the same original paper about super(edge)- magic graphs[8] Enomoto et al conjectured that all trees are super(edge) magic. This conjecture has become already a classical problem in the world of graph labelings. However in spite of all efforts, the lack of general results is still obvious when consulting the literature. This is not surprising. Since Similar things happen with other well known conjectures involving trees and labelings. For instance, the graceful tree conjecture, which states that all trees are graceful, is a 35 years old conjecture that is still very far away from a complete solution. A similar thing happens with the harmonious tree conjecture [5] , which states that all trees are harmonious. In fact all these labeling conjectures are all related among them and the solution to any one of them would through a lot of light into the solution of the other conjectures. The interested reader about relations among labeling can find in [2,7,9,11]an excellent information source.

For every graph G, Kotzig and Rosa [14] proved that there exists an edge magic graph H such that $H \cong G \cup nK_1$, for some non negative integer n. This motivated them to define the edge magic deficiency of a graph. The edge magic deficiecy $\mu(G)$, of a graph G is the smallest non negative integer n for which the graph $G \cup nK_1$, as edge- magic. In [10],the authors analogously defined the concept of super edge magic (In [3] super magic) deficiency $\mu_s(G)(\mu_i(G))$ of a graph G to be either the smallest non negative integer n with the property that the graph $G \cup nK_1$ is super edge magic(Super Magic) or $+\infty$ if there exists no such integer n. It follows immediately that $\mu(G) \leq \mu_s(G)(\mu(G) \leq \mu_i(G))$ for any graph G.

To present the new results contained in this paper, the following lemmas taken from [2] will prove to be useful.

DefinitionThe edge magic deficiency of a graph G, $\mu(G)$ is defined as $\mu(G) = min\{n \geq 0; G \cup nK_1\}$ is an edge-magic graph.

Lemma 1.1. *A graph G of order p and size q is super magic if and only if there exists a bijectiove function* $f : V(G) \rightarrow \{q+1, q+2, - - -, q+p\}$ *such that the set*

$$S = \{\lambda - [f(u) + f(v)] : uv \in E(G)\}$$

consists of q consecutive integers. In such a case f extends to a super magic labeling of G with valence $\lambda = q + s$ *where* $s = min(S)$ *and* $S = \{f(u) + f(v)|uv \in E(G)\} = \{\lambda - 1, \lambda - 2, - - -, \lambda - q\}$.

Due to lemma 1.1 it is sufficient to exhibit the vertex labeling of a super magic graph, however we will provide the valence to increase the clarity of our results.

We will also utilize the following lemma from [2]

Lemma 1.2. *Let G be a super magic (p,q) graph and f be a super magic labeling of G. Then* $\sum f(v)degv = qs + \binom{q}{2}$ *where s is defined in the previous lemma. A graph G is said to be an even graph if all of its vertices have even degree. Thus with this definition in hand we are able to state one more technical lemma found in [3].*

Lemma 1.3. *let G be a graph of size q having the property that for all non empty sets* V_1 *and* V_2 *such that* $V_1 \cup V_2 = V(G)$ *and* $V_1 \cap V_2 = \phi$

$|\{uv \in E(G) : u \in V_1 and\ v \in V_2\}|$ *is neither* $\lfloor q/2 \rfloor$ *nor* $\lceil q/2 \rceil$ *.Then* $\mu_i(G) = +\infty$

The above lemma shows that adding a finite number of isolated vertices to a graph that is not super magic need not produce a super magic graph. This was unexpected in light of the fact that Kotzig and Rosa [13] proved that every graph has finite edge- magic deficiency $\mu(G)$*. Indeed it is easy to construct a graph G such that* $\mu(G) - \mu_i(G) = +\infty$ *(See [3] for example)*

Actually more is true as given a non negative integer n it is always possible to construct a graph G such that $\mu(G) - \mu_i(G) = n$*. Kotzig and Rosa [14] proved that all complete*

bipartite graphs are edge- magic, which implies that $\mu(K_{2,n+1}) = 0$. *On the other hand , Akka, et al showed that in [3] that* $\mu_i(K_{2,n+1}) = n$.

To conclude this introduction we state a result by the authors [16] and state and prove a corollary to it which will later serve as the basis for some remarks and conjectures.

Lemma 1.4. *If G is a complementary edge magic (super magic) bipartite or tripartite graph and m is odd then mG is complementary edge magic (super magic).*

Lemma 1.5. *If G is a bipartite or tripartite graph and m is odd then* $\mu(G) \leq m\mu(G)$ $\mu_s(G) \leq m\mu_s(G)$ *and* $\mu_i(G) \leq m\mu_i(G)$. *Where* $\mu_s(G)$ *denotes Super edge magic deficiency.*

Proof. let G be a bipartite or tripartite graph and m be odd. First notice that if $\mu_i(G) = +\infty$, then the inequality for μ_i is trivial. Hence without loss of generality we may assume that $\mu_i(G) < +\infty$. Moreover as we commented above, it is an established fact that $\mu(G) < +\infty$ for any graph G. The remainder of the proof is identical for either μ, μ_s or μ_i and if G is bipartite or tripartite. Now there exists a non-negative integer n such that $G \cup nK_1$ is super magic. Further more, $G \cup nK_1$ is certainly bipartite. Therfore $m(G \cup nK_1) \cong mG \cup mnK_1$ is super magic by the previous theorem which implies that $\mu_i(G) \leq m\mu_i(G)$. □

2. Main Results on Forests

In [3], D G Akka et al give a constructive proof that the super magic deficiency of forest is finite. That proof is too involed to easily glean from it an upper bound for $\mu_i(F)$ when F is a forest, but such a bound would certainly be a large quantity relative to the order of F. However, the authors believe that $\mu_i(F)$ is always a small number. The evidence for such a belief come from the authors previous explorations on the super magic properties of forests [3, 4, 5, 6] and are bolstered by the results included in this section.

Now, with the aid of the super magic characterization of the forest $K_{1,1} \cup K_{1,n}$ found in [6] we are able to provide the following theorem.

Theorem 2.1. *for every positive integer n*

$$\mu_i(P_2 \cup K_{1,n}) = \begin{cases} 0, & \textit{if } n \textit{ is even ;} \\ 1, & \textit{if } n \textit{ is odd;} \end{cases}$$

Proof. First notice that the forest $P_2 \cup K_{1,n} \cong K_{1,1} \cup K_{1,n}$ is shown to be super magic in [6] if and only if n is even. Consequently it is sufficient to verify that $\mu_i(P_2 \cup K_{1,n}) \leq 1$ when n is odd.

Now assume that n is odd and let $F \cong P_2 \cup K_{1,n} \cup K_1$ be the forest with $V(F) = \{u, w, x, y\} \cup \{v_i | 1 \leq i \leq n\}$ and $E(E) = \{uv_i | 1 \leq i \leq n\} \cup \{xy\}$. Then the vertex labeling $f : V(F) \to \{n+2, n+3, - - -, (2n+5)\}$ such that $f(x) = 2n+5, f(y) = n+2, f(u) = (3n+7)/2$

$$f(v_i) = \begin{cases} 2n+5-i, & \text{if } i \in [1, \frac{n+1}{2}]; \\ 2n+4-i, & \text{if } i \in [\frac{n+1}{2}+1, n]. \end{cases}$$

and $f(w) = n+3$ extends to a super magic labeling of F with valence $\frac{7n+17}{2}$.

Therfore $\mu_i(P_2 \cup K_{1,n}) \leq 1$ when n is odd which completes the proof.

The super magic characterization of the forest $K_{1,2} \cup K_{1,n}$ found in [6] is extended in the following theorem.

Theorem 2.2. *For every positive integer n*

$$\mu_i(P_3 \cup K_{1,n}) = \begin{cases} 0, & \textit{if } n \equiv 0 \pmod 3 \textit{ ;} \\ 1, & \textit{otherwise} \end{cases}$$

Proof. First notice that the forest $P_3 \cup K_{1,n} \cong K_{1,2} \cup K_{1,n}$ is known to be super magic if and only if $n \equiv 0 \pmod 3$ see [6]. Hence in the remainder of the proof, assume that $n \not\equiv 0 \pmod 3$, and thus our goal is to show that in such a case $\mu_i(P_3 \cup K_{1,n}) \leq 1$.

To do this , let $F \cong P_3 \cup K_{1,n} \cup K_1$ be the forest with $V(F) = \{u, w, x, y, z\} \cup \{v_i | 1 \leq i \leq n\}$ and

$E(F) = \{uv_i | 1 \leq i \leq n\} \cup \{xy, xz\}$ and consider two cases

Case1. Let n be odd and define $f : V(F) \rightarrow \{n+3, n+4, - - -, 2n+7\}$ to be the vertex labeling such that $f(x) = 2n+7, f(y) = n+4, f(z) = n+3, f(u) = \frac{3n+11}{2}$

$$f(v_i) = \begin{cases} 2n+7-i, & \text{if } i \in [1, \frac{n+1}{2}]; \\ 2n+5-i, & \text{if } i \in [\frac{n+1}{2}+1, n] \end{cases}$$

And $f(w) = \frac{3n+9}{2}$.

Then by lemma 1.1, f extends to a super magic labeling of F with valence $\frac{7n+25}{2}$.

Case2. Let n be even and define $f : V(F) \rightarrow \{n+3, n+4, - - -, 2n+7\}$ to be the vertex labeling such that $f(x) = 2n+7, f(y) = n+4, f(z) = n+3, f(u) = \frac{3n+10}{2}$.

$$f(v_i) = \begin{cases} 2n+7-i, & \text{if } i \in [1, n/2]; \\ 2n+5-i, & \text{if } i \in [\frac{n}{2}+1, n] \end{cases}$$

And $f(w) = \frac{3n+12}{2}$.

Then , the lemma 1.1, f extends to a super magic labeling of F with valence $\frac{7n+24}{2}$.

Therefore the proof is completed.

By labeling the vertices of degree one of the forest $F \cong P_1 \cup K_{1,n}$ with the n positive integers $n+3, n+4, - - -2n+2$, the isolated vertex of F with n+2 and the remaining vertex n+1. We obtain a super magic labeling of F with valence 3n+4.

Therefore by the previous two results together with the fact presented in [6] that the forest $P_m \cup K_{1,n}$ is super magic for every two integers $m \geq 1$ and $n \geq 1$, we immediately obtain the following theorem. □

Theorem 2.3. *For every two positive integers m and n*

$$\mu_i(P_m \cup K_{1,n}) = \begin{cases} 1, & \textit{if m=2 and n is odd or m=3 and } n \not\equiv 0 \pmod 3 \textit{ ;} \\ 0, & \textit{otherwise} \end{cases}$$

Proof. Now, we study the super magic properties of the forest $K_{1,m} \cup K_{1,n}$.

For this purpose, let $F \cong K_{1,m} \cup K_{1,n} \cup K_1$ be the forest with $V(F) = \{v_i | 1 \leq i \leq m+n+3\}$ and $E(F) = \{V_2V_i | 3 \leq i \leq m+2\} \cup \{v_1v_i | m+4 \leq i \leq m+n+3\}$ Then vertex labeling $f : V(F) \to \{m+n+1, m+n+2, - - -, 2m+2n+3\}$ such that $f(v_i) = 2m+2n+4-i\,(1 \leq i \leq m+n+3)$ induces a super magic labeling of F with valence $4\,(m+n+1)$. Thus $\mu_i(K_{1,m} \cup K_{1,n}) \leq 1$ for every positive integer n.

On other hand, Ivanco and Luckanicova proved in [13] that $K_{1,m} \cup K_{1,n}$ is super edge magic if and only if either m is a multiple of n+1 or n is a multiple of m+1. Combining this with the above fact, we have the following theorem. □

Theorem 2.4. *For every two positive integers m and n*

$$\mu_i(k_{1,n} \cup K_{1,n}) = \begin{cases} 0, & \textit{either m is a multiple of n+1 or n is a multiple of m+1;} \\ 1, & \textit{otherwise.} \end{cases}$$

We now inquire into the super magic deficiency of the forest $P_m \cup P_n$. By theorem 2.4, observe that $\mu_i(2P_n) = 1$ for n = 2 or 3 . On the other hand, the authors proved in [5] that the forest $P_m \cup P_n$ is super magic if and only if $(m,n) \notin \{(2,2)(3,3)\}$. Thus we obtain the following theorem.

Theorem 2.5. *For all positive integers m and n*

$$\mu_i(P_m \cup P_n) = \begin{cases} 1, & \textit{if } (m,n) \in \{(2,2)(3,3)\}\,; \\ 0, & \textit{otherwise.} \end{cases}$$

Now we endeover to find parallel complementary edge magic deficiency results to the ones on super magic deficiency presented earlier in this section. This is interesting as Kotzig [15] proved that there are infinitely many forests that are not complementary edge magic. To do so we start with the following corollary.

Corollary2.6. *For every two positive integer m and n*

$$\mu_i(K_{1,m} \cup K_{1,n}) = \begin{cases} 0, & \textit{if mn is even;} \\ 1, & \textit{if mn is odd.} \end{cases}$$

Proof. First by theorem 2.4 we have that
$\mu_i(K_{1,m} \cup K_{1,n}) \leq \mu_i(K_{1,m} \cup K_{1,n}) \leq 1$ Second, notice that the Sanjay Roy and Akka in [16] proved that the forest $K_{1,m} \cup K_{1,n}$ is complementary edge magic if and only if mn is even.

If we let n=1 and 2 in the above theorem then we can compute the complementary edge magic deficiencies of the forests $K_{1,1} \cup K_{1,n} \cong P_2 \cup K_{1,n}$ and $K_{1,2} \cup K_{1,n} \cong P_3 \cup K_{1,n}$ respectively. By adding these facts to theorem 2.3 we have the following corollary. □

Corollary2.7. *For any two positive integers m and n*

$$\mu_i(P_m \cup K_{1,n}) = \begin{cases} 1, & \textit{if m=2 and n is odd;} \\ 0, & \textit{otherwise.} \end{cases}$$

Now by the above corollary, $2P_3$ is complementary edge magic and $2P_2$ is not. Therefore by theorem 2.5, we have the following corollary.

$$\mu_i(P_m \cup P_n) = \begin{cases} 1, & \textit{if (m,n)=(2,2);} \\ 0, & \textit{otherwise.} \end{cases}$$

The following two conjectures are born out of the results in this section and those found in [3,4,5,6]. First we state the weaker of the two.

Conjecture2.8. *If F is a forest with two components then* $\mu_i(F) \leq 1$

In fact we believe that more is true.

Conjecture2.9. *If F is a forest with two components then* $\mu_i(F) \leq 1$

3. Main results on 2-regular graphs

A recurrent object of interest for the authors are the super magic properties of 2-regular graphs, for example see[4,5,6]. Indeed, a long term goal has become the evincing of the super magic deficiencies of all of these graphs. Therfore the results in this section are intended to narrow the gap between what is known and our objective.

Now consider the following result found in [4]

Theorem3.1 *The 2- regular graph mC_n is super magic if and only if m and n are odd.*

The above theorem allows us to compute the super magic deficiency of the 2-regular graph $2C_n$.

Theorem3.2 *For every integer $n \geq 3$.*

$$\mu_i(2C_n) = \begin{cases} 1, & \text{if } n \text{ is even;} \\ +\infty, & \text{if } n \text{ is odd.} \end{cases}$$

Proof : *First assume that n is odd, then by lemma 1.3 we obtain that $\mu_i(2C_n) = +\infty$.*

Next, assume that n is even, then by theorem 3.1, the 2-regular graph $2C_n$ is not super magic. Hence $\mu_i(2C_n) \geq 1$.

To show that $\mu_i(2C_n) \leq 1$, let $G \cong 2C_n \cup K_1$ be the graph with
$V(G) = \{x_i | 1 \leq i \leq n\} \cup \{y_i | 1 \leq i \leq n\} \cup \{z\}$ *and*
$E(G) = \{x_i y_i | 1 \leq i \leq n\} \cup \{x_i y_{i+1} | 1 \leq i \leq \frac{n}{2} - 1\} \cup \{x_{n/2} y_1\} \cup \{x_i y_{i+1} | \frac{n}{2} + 1 \leq i \leq n-1\} \cup \{x_n y_{\frac{n}{2}+1}\}$ *and consider two cases.*

Case1. *For $n = 4k$ where k is a positive integer, define the vertex labeling*
$f : V(G) \to \{8k+1, 8k+2, - - -, 16k\}$ *such that*

$$f(x_i) = \begin{cases} 12k - i, & \text{if } i \in [1, k]; \\ 8k + i - 1, & \text{if } i \in [k+1, 2k]; \\ 17k - i, & \text{if } i \in [2k+1, 3k-1]; \\ 16k - 1, & \text{if } i \in [3k, 4k]. \end{cases}$$

$$f(y_i) = \begin{cases} 16k + 1 - i, & \text{if } i \in [1, k+1]; \\ 12k + i - 1, & \text{if } i \in [k+2, 2k]; \\ 8k + i - 1, & \text{if } i \in [2k+1, 3k]; \\ 12k - i, & \text{if } i \in [3k+1, 4k]. \end{cases}$$

And $f(z) = 14k$

Case2. *For* $n = 4k + 2$*, where* k *is a positive integer, define the vertex labeling.* $f : V(G) \to \{8k + 5, 8k + 6, - - -, 16k + 9\}$ *such that*

$$f(x_i) = \begin{cases} 10k - 5 + i, & \text{if } i \in [1, k-1]; \\ 10k + 6 - i, & \text{if } i \in [k, 2k+1]; \\ 18k + 1 - i, & \text{if } i \in [2k+2, 3k+2]; \\ 10k + 5 + i, & \text{if } i \in [3k+3, 4k+2]. \end{cases}$$

$$f(y_i) = \begin{cases} 14k + 8 + i, & \text{if } i \in [1, k]; \\ 14k + 8 - i, & \text{if } i \in [k+1, 3k+3]; \\ 6k + 3 + i, & \text{if } i \in [3k+4, 4k+2]. \end{cases}$$

And $f(z) = 14k + 8$

Therefore, by lemma 1.1, f *extends to a super magic labeling of* G *with Valence* $7n - 1$.

With the aid of theorem 3.1, we are now able to proved the super magic deficiency of the 2-regular graph $3C_n$.

Theorem3.3 *For every integers* $n \geq 3$.

$$\mu_i(3C_n) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \equiv 0 \pmod 4; \\ +\infty, & n \equiv 2 \pmod 4. \end{cases}$$

Proof : *If follows from theorem 3.1 that* $\mu_i(3C_n) = 0$ *when* n *is odd so we now suppose that* $n \equiv 0 \pmod 4$. *Then by theorem 3.1, the 2-regular graph* $3C_n$ *is not super magic that is* $\mu_i(3C_n) \geq 1$.

To, verify that $\mu_i(3C_n) \leq 1$, *let* $G \cong 3C_n \cup K_1$, *be the graph with*
$V(G) = \{x_i | 1 \leq i \leq 3n/2\} \cup \{y_i | 1 \leq i \leq 3n/2\} \cup \{z\}$
and $E(G) = \{x_i y_i | 1 \leq i \leq 3n/2\} \cup \{x_i y_{i+1} | 1 \leq i \leq \frac{n}{2} - 1\} \cup \{x_{n/2} y_1\} \cup \{x_i y_{i+1} | \frac{n}{2} + 1 \leq i \leq n - 1\} \cup \{x_n y_{\frac{n}{2}+1}\} \cup \{x_i y_{i+1} | n + 1 \leq i \leq \frac{3n}{2} - 1\} \cup \{x_{3n/2} y_{n+1}\}$

and consider two cases.

Case1. *For n=4, define the vertex labeling*

$f : V(G) \to \{13, 14, 15, - - -, 25\}$ *such that*

$(f(x_i))_{i=1}^{6} = \{25, 23, 24, 19, 22, 21\}$

$(f(y_i))_{i=1}^{6} = \{18, 17, 20, 14, 15, 13\}$

$f(z) = 16$

Case2. *For $n = 4k$ where k is an integer with $k \geq 2$, define the vertex labeling*

$f : V(G) \to \{12k + 1, 12k + 2, - - - -, 24k + 1\}$ *such that* $f(x_i) = 6n+2-i$ *if* $i \in [1, 6k]$

$$f(y_i) = \begin{cases} 16k + 3, & \text{if } i = 1; \\ 18k - i + 3, & \text{if } i \in [2, k]; \\ 15k + 2, & \text{if } i = k + 1; \\ 18k - i + 4, & \text{if } i \in [k + 2, 2k]; \\ 18k - i + 2, & \text{if } i \in [2k + 1, 3k - 1]; \\ 18k - i - 1, & \text{if } i \in [3k, 4k - 1]; \\ 16k + 2, & \text{if } i = 4k; \\ 18k - i, & \text{if } i \in [4k + 1, 5k]; \\ 15k, & \text{if } i = 5k + 1; \\ 18k - i + 1, & if i \in [5k + 2, 6k]; \end{cases}$$

Thus by lemma 1.1, f extends to a super magic labeling of G with valence $\frac{21n}{2} + 3$ *and we conclude that* $\mu_i(3C_n) = 1$ *when* $n \equiv 0 \pmod 4$. *Finally the remaining case immediately follows from lemma 1.3.*

Now in light of the above results it seems natural to explore the super magic deficiency of the 2- regular gaph $4C_n$. *However , the authors have only been able to provide a partial solution to this question which is contained in the following theorem and the subsequent comment.*

Theorem3.4 *For every positive integer* $n \equiv 0 \pmod 4$, $\mu_i(4C_n) = 1$.

Proof : *Throughout this proof, suppose that* $n \equiv 0 \pmod 4$. *Then notice by theorem 3.1, that the 2-regular graph* $4C_n$ *is not super magic. Thus it suffices to show that* $\mu_i(4C_n) \leq 1$. *To do this, let* $G \cong 4C_n \cup K_1$ *be the graph with*

$V(G) = \{x_i | 1 \leq i \leq 2n\} \cup \{y_i | 1 \leq i \leq 2n\} \cup \{z\}$

and $E(G) = \{x_i y_i | 1 \leq i \leq n\} \cup \left\{x_i y_{i+1} | 1 \leq i \leq \frac{n}{2} - 1\right\} \cup \left\{x_{\frac{n}{2}} y_1\right\} \cup \left\{x_i y_{i+1} | \frac{n}{2} + 1 \leq i \leq n-1\right\} \cup \left\{x_n y_{\frac{n}{2}+1}\right\} \cup \left\{x_i y_{i+1} | n+1 \leq i \leq \frac{3n}{2} - 1\right\} \cup \left\{x_{\frac{3n}{2}} y_{n+1}\right\} \cup \left\{x_i y_{i+1} | \frac{3n}{2} + 1 \leq i \leq 2n-1\right\} \cup \left\{x_{2n} y_{\frac{3n}{2}+1}\right\}$, *and consider five cases according to the possible values for the integer n.*

Case1. *For* $n = 4$, *define the vertex labeling* $f : V(G) \to \{17, 18, - - -, 33, \}$ *such that*

$(f(x_i))_{i=1}^{8} = \{33, 31, 32, 28, 30, 26, 27, 25\}$

$(f(y_i))_{i=1}^{8} = \{24, 23, 21, 19, 22, 20, 18, 17\}$

$f(z) = 29$

Case2. *For* $n = 8$, *define the vertex labeling* $f : V(G) \to \{33, 34, - - -65\}$ *such that*

$(f(x_i))_{i=1}^{16} = \{65, 64, 62, 61, 59, 56, 52, 51, 63, 60, 58, 54, 55, 53, 49, 50\}$

$(f(y_i))_{i=1}^{16} = \{48, 47, 46, 45, 44, 43, 38, 34, 42, 41, 40, 39, 37, 36, 35, 33\}$

$f(z) = 57$

Case3. *For* $n = 12$, *define the vertex labeling* $f : V(G) \to \{49, 50, - - -, 98\}$ *such that*

$(f(x_i))_{i=1}^{24} = \{97, 96, 95, 94, 93, 87, 92, 91, 83, 79, 78, 75, 90, 89, 88, 86, 84, 81, 82, 80, 76, 73, 74, 77\}$

$(f(y_i))_{i=1}^{24} = \{72, 71, 70, 69, 68, 67, 66, 65, 64, 57, 53, 50, 63, 62, 61, 60, 59, 58, 56, 55, 54, 51, 49, 52\}$

$f(z) = 85$

Case4. *For* $n = 8k + 8$ *where* k *is a positive integer, suppose that* l *is an integer with* $1 \leq l \leq k$ *and define the vertex labeling*

$f : V(G) \to \{32k + 33, 32k + 34, - - -, 64k + 65\}$ *such that*

$$f(x_i) = \begin{cases} 64k + 66 - i, & \text{if } i \in [1, 4k+3]; \\ 64k + 67 - i, & \text{if } i \in [4k+5, 6k+5]; \end{cases}$$

$f(x_{4k+4}) = 56k + 59$

$f(x_{6k+6}) = 54k + 56$

$$f(x_{8k+7}) = 49k + 51$$
$$f(x_{8k+8}) = 49k + 50$$
$$f(x_{10k+11}) = 56k + 58$$
$$f(x_{12k+12}) = 54k + 54$$
$$f(x_{12k+13}) = 54k + 55$$
$$f(x_{12k+14}) = 54k + 53$$
$$f(x_{16k+15}) = 54k + 49$$
$$f(x_{16k+16}) = 54k + 50$$

$$f(x_i) = \begin{cases} 55k - 6l + 57, & \textit{if } i = 6k + 2l + 5; \\ 55k - 6l + 56, & \textit{if } i = 6k + 2l + 6; \\ 56k - 2l + 58, & \textit{if } i = 10k + 2l + 10; \\ 56k - 2l + 57, & \textit{if } i = 10k + 2l + 11; \\ 60k - 6l + 48, & \textit{if } i = 10k + 2l + 15; \\ 60k - 6l + 47, & \textit{if } i = 10k + 2l + 16; \\ 48k + 6l + 43, & \textit{if } i = 14k + 2l + 13; \\ 48k + 6l + 44, & \textit{if } i = 14k + 2l + 14; \end{cases}$$

$$f(x_{8k+i+8}) = 59k - i + 61 \ \textit{if } i \in [1, 2k + 2]$$
$$f(y_i) = 48k - i + 49 \ \textit{if } i \in [1, 6k + 6]$$
$$f(y_{6k+7}) = 38k + 38$$
$$f(y_{8k+8}) = 32k + 34$$
$$f(y_{12k+13}) = 38k + 37$$
$$f(y_{12k+14}) = 38k + 36$$
$$f(y_{12k+15}) = 38k + 35$$
$$f(y_{14k+16}) = 32k + 33$$

$$f(y_i) = \begin{cases} 38k - 6l + 40, & \text{if } i = 6k + 2l + 6; \\ 38k - 6l + 39, & \text{if } i = 6k + 2l + 7; \\ 38k - 6l + 36, & \text{if } i = 12k + 2l + 14; \\ 38k - 6l + 35, & \text{if } i = 12k + 2l + 15; \\ 32k + 6l + 31, & \text{if } i = 12k + 2l + 17; \\ 32k + 6l + 32, & \text{if } i = 12k + 2l + 18; \end{cases}$$

$f(y_{8k+i+8}) = 42k - i + 43$ *if* $i \in [1, 4k + 4]$ *and* $f(z) = 56k + 57$.

Case5. *For* $n = 8k + 12$ *where* k *is a positive integer, assume that* l *is an integer with* $1 \leq l \leq k$ *and define the vertex labeling.* $F : V(G) = \{32k + 49, 32k + 50, --, 64k + 97\}$ *such that*

$$f(x_i) = \begin{cases} 64k + 98 - i, & \text{if } i \in [1, 4k + 5]; \\ 64k + 99 - i, & \text{if } i \in [4k + 7, 6k + 8]; \end{cases}$$

$f(x_{4k+6}) = 56k + 87$
$f(x_{6k+9}) = 54k + 83$
$f(x_{8k+10}) = 48k + 79$
$f(x_{8k+11}) = 48k + 78$
$f(x_{8k+12}) = 48k + 75$
$f(x_{10k+16}) = 56k + 86$
$f(x_{12k+18}) = 54k + 81$
$f(x_{12k+19}) = 54k + 82$
$f(x_{12k+20}) = 54k + 80$
$f(x_{14k+21}) = 48k + 76$
$f(x_{14k+22}) = 48k + 73$
$f(x_{14k+23}) = 48k + 74$

$$f(x_{14k+24}) = 48k + 77$$

$$f(x_i) = \begin{cases} 54k - 6l + 85, & \textit{if } i = 6k + 2l + 8; \\ 54k - 6l + 84, & \textit{if } i = 6k + 2l + 9; \\ 54k - 6l + 81, & \textit{if } i = 12k + 2l + 9; \\ 54k - 6l + 80, & \textit{if } i = 12k + 2l + 10; \\ 48k + 6l + 76, & \textit{if } i = 14k + 2l + 23; \\ 48k + 6l + 77, & \textit{if } i = 14k + 2l + 24; \end{cases}$$

$$f(y_{8k+i+12}) = \begin{cases} 58k + 91 - i, & \textit{if } i \in [1, 2k + 3]; \\ 58k + 89 - i, & \textit{if } i \in [2k + 5, 4k + 5]; \end{cases}$$

$$f(y_i) = 48k - i + 73 \ \textit{if } i \in [1, 6k + 9]$$

$$f(y_{6k+10}) = \begin{cases} 58k - i + 91, & \textit{if } i \in [1, 2k + 3]; \\ 58k - i + 89, & \textit{if } i \in [2k + 5, 4k + 5]; \end{cases}$$

$$f(y_i) = 48k - i + 73 \ \textit{if } i \in [1, 6k + 9]$$
$$f(y_{6k+10}) = 38k + 57$$
$$f(y_{8k+11}) = 32k + 53$$
$$f(y_{8k+12}) = 32k + 50$$
$$f(y_{12k+19}) = 38k + 56$$
$$f(y_{12k+20}) = 38k + 55$$
$$f(y_{12k+21}) = 38k + 54$$
$$f(y_{14k+22}) = 32k + 51$$

$f(y_{14k+23}) = 32k + 49$
$f(y_{14k+24}) = 32k + 52$

$$f(y_i) = \begin{cases} 38k - 6l + 59, & \text{if } i = 6k + 2l + 9; \\ 38k - 6l + 58, & \text{if } i = 6k + 2l + 10; \\ 38k - 6l + 55, & \text{if } i = 12k + 2l + 20; \\ 32k - 6l + 54, & \text{if } i = 12k + 2l + 21; \\ 32k + 6l + 50, & \text{if } i = 14k + 2l + 23; \\ 32k + 6l + 51, & \text{if } i = 14k + 2l + 24; \end{cases}$$

$f(y_{8k+i+12}) = 42k - i + 64$ *if* $i \in [1, 4k + 6]$ *and* $f(z) = 56k + 85$.

Therefore by lemma 1.1, f extends to a super magic labeling of G with valence $14n + 2$ *which leads to conclude that* $\mu_i(4C_n) = 1$.

When $n \not\equiv 0 \pmod 4$ *our knowledge is limited to the value of* $\mu_i(4C_6)$ *which we determine to be 1 as follows: First by theorem 3.1, the 2-regular graph* $4C_6$ *is not super magic, however, the graph* $4C_6 \cup k_1$ *is super magic by labeling the vertices in its cycles with 12-25-11-24-10-21-12; 9-23-7-22-5-16-9; 8-20-6-18-4-15-8 and 3-17-1-13-2-14-3; and its isolated vertex 19 to obtain a valence of 86.*

Now, lemma 1.3, all the results in this section and the author's computation of $\mu_i(C_n)$ *in [3] lead us to the following conjecture.*

Conjecture3.5 *For every two integer* $m \geq 1$ *and* $n \geq 3$.

$$\mu_i(mC_n) = \begin{cases} 0, & \text{if } mn \text{ is odd;} \\ 1, & \text{if } mn \equiv 0 \pmod 4; \\ +\infty, & \text{if } mn \equiv 2 \pmod 4. \end{cases}$$

References

[1] D G Akka and N.S Warad, Super magic strength of a graph, Indian J pure and Appl. math. 41(4)(2010) 557-568

[2] D G Akka Some properties of Super magic labeling of graph(pre-print).

[3] D G Akka and N S Warad, On Super magic deficiency of graphs (doctoral Thesis of N S Warad 2012)

[4] D G Akka and Dayanand G. K., The Super magic properties connected and disconnected graphs(pre-print)

[5] D G Akka and Dayanand G. K., The Super magic properties of 2-regular graphs with two components(pre-print)

[6] D G Akka and Dayanand G. K., Of Super magic labeling of forests(pre-print)

[7] G Chartrand and L. Lesniak, Graphs and Daigraphs, second edition, Wadsworth and books/ cole advanced books and software Monterey (1986).

[8] H Enomoto, A Llado, T. Nakamigawa and G Ringel Super edge magic graphs, SUT J. Math. 34 (1998) 105-109.

[9] R. M. Figueroa- Centeno, R. Ichishima and F.A. Muntaner Batle, The place of super edge magic labelings among other classes of labelings, Discrete math. 231 (1-3)(2001)153-168.

[10] R. M. Figueroa- Centeno, R. Ichishima and F.A. Muntaner Batle, On the super edge magic deficiency of graphs,Ars Combin.,78 (2006)33-45.

[11] J. A. Gallian, A dynamic survey of graph labeling Electron J. Combin., 5(2004) # DS6.

[12] R. L. Graham and N.J.A. Sloane, On additive bases and harmonious graphs SIAM J.Alg.,Discrete math., 11(1980)382-404.

[13] J. Ivanco and Luckanicova, On edge magic disconnected graphs, SUT J. of math. 38 (2002) 175-184.

[14] A Kotzig and A Rosa, magic valuations of finite graphs, Canad math Bull. 13(1970)451-461.

[15] A Kotzig On a class of graphs with magic valuations, Publ. des Centre de Rechercher Mathematiques universite de Montreal 136(1971).

[16] Sanjay Roy and D G Akka, On complementary edge magic labeling of certain Graphs, American J. of Computional and Applied mathematics, Vol. 1, No.1(2011)1-5.

Mallikarjun Ghaleppa, Department of Mathematics, Sri Krishna Institute of Technology, Bengaluru – 560090, INDIA

E-mail address: mgdsagar@gmail.com

Danappa G Akka, Department of Mathematics, Sri Revanasiddeswara Institute of Technology, Bengaluru – 562157, INDIA

E-mail address: drdgakka1233@yahoo.com