ON HARMONIOUS COLORING OF $M(S_n)$ AND $M(D^m_3)$

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Abstract: A harmonious coloring is a proper vertex coloring in which every pair of colors appears on at most one pair of adjacent vertices.

In this paper, harmonious coloring of $M(S_n)$ and $M(D^m_3)$ are studied. Some structural properties of them are discussed. Also, their harmonious chromatic number was obtained.

Keywords: Graph coloring, middle graph, sunlet graph, dutch–windmill graph, harmonious coloring and harmonious chromatic number.

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1. Introduction

Let G be a finite, undirected graph with no loops and multiple edges. The graph G has the vertex set V(G) and the edge E(G). Graph coloring is coloring of G such that no two adjacent vertices share the same color. A harmonious coloring [2, 3, 4, 6] is a proper vertex coloring in which every pair of colors appears on at most one pair of adjacent vertices. The harmonious chromatic number is the minimum number of colors needed for any harmonious coloring of G.

The middle graph of G denoted by M(G), is defined as follows: The Vertex set of M(G) is V(G) U E(G) in which two elements are adjacent in M(G) if the following conditions hold.

(i) $x, y \in E(G)$ and they are adjacent in G.

(ii) $x \in V(G)$, $y \in E(G)$ and y is incident on x in G.
A Sunlet graph \( (S_n) \) on \( 2n \) vertices is a graph obtained by attaching \( n \) pendent edges to the cycle graph \( C_n \). The Dutch-windmill graph \( (D_3^m) \), also called a friendship graph, is obtained by taking \( m \) copies of the cycle \( C_3 \) with a vertex in common.

2. **Structural properties of middle graph of Sunlet Graph and Dutch-Windmill Graph**

- Number of vertices in \( M(S_n) \), \( p = 4n \)
- Number of edges in \( M(S_n) \), \( q = 7n \)
- Maximum degree in \( M(S_n) \), \( \Delta = 6 \)
- Minimum degree in \( M(S_n) \), \( \delta = 1 \)
- Number of vertices in \( (D_3^m) \), \( = 5m + 1 \)
- Maximum degree in \( (D_3^m) \), \( \Delta = 2m + 2 \)
- Minimum degree in \( (D_3^m) \), \( \delta = 2 \)

3. **Harmonious coloring of \( M(S_n) \) and \( M(D_3^m) \)**

**Theorem 3.1**: For the middle graph of Sunlet graph \( M(S_n) \), the harmonious chromatic number is

\[
\chi_H [M(S_n)] = \begin{cases} 
3n + 1 & \text{if } n \text{ is odd} \\
3n & \text{if } n \text{ is even}
\end{cases}
\]

**Proof**: Let \( (S_n) \) be the Sunlet graph on ‘2n’ vertices with \( n \) pendent edges to the cycle graph \( C_n \). Let \( u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n \) be the vertices of middle graph of sunlet graph (taken in clock wise order).

By definition of middle graph, each edge of the Sunlet Graph is subdivided by a new vertex, assume that each edge \( (v_i, v_{i+1}), (u_i, u_{i+1}), (u_{n,1}) \) and \( (v_j, u_j) \)
is subdivided by the vertex \( v_{i,i+1}, u_{i,i+1}, u_{n,1} \) and \( e_j \) for \( 1 \leq i \leq n, 1 \leq j \leq n \) respectively.

Clearly \( V[M(S_n)] = \{v_i\} \cup \{u_i\} \cup \{v_{i,i+1}\} \cup \{u_{i,i+1}\} \cup \{u_{n,1}\} \cup \{e_j\} \)

Now we assign the colors to the vertices of \( M(S_n) \) as follows:

Consider a color class \( C = \{c_1, c_2, c_3, \ldots, c_n, c_{2n}, c_{2n+1}, \ldots, c_{3n}, c_{3n+1}\} \)

**Case (i): If \( n \) is odd**

Assign the color \( c_i \) to \( v_i \) and \( c_{i+2n+1} \) to \( u_i \) for \( 1 \leq i \leq n \)

Assign the color \( c_{(n+i)+1} \) to \( u_{i,i+1} \) and color \( c_{2n+1} \) to \( u_{n,1} \)

**Case (ii): If \( n \) is even**

color \( c_i \) to \( v_i \) and color \( c_{n+i} \) to \( u_i \) for \( 1 \leq i \leq n \)

color \( c_{2n+i} \) to \( u_{i,i+1} \) and at last assign the color \( c_{3n} \) to \( u_{n,1} \)

and \( c_{i+1} \) to \( e_i \) for \( 1 \leq i \leq n-1 \)

It is clearly verified that the above said coloring is harmonious and is minimum.

Hence \( \chi_H[M(S_n)] = \begin{cases} 3n+1 & \text{if } n \text{ is odd} \\ 3n & \text{if } n \text{ is even} \end{cases} \)

**Fig.1: Sunlet Graph \( S_3 \) and \( S_4 \)**
Theorem 3.2: For the Dutch-wind mill graph $D_3^m$, $\chi_H [M(D_3^m)] = 3m + 3, n \geq 3$

Proof: Consider the Dutch-wind mill graph $D_3^m$ formed from $m$ copies of the cycle $C_3$ with vertices, $\{u_1^1, u_2^i, u_3^i\}$, $i = 1, 2, 3 \ldots m$ with the vertex $u_1^i$ in common.

(i.e) $u_1^1 = u_1^2 = u_3^3 \ldots = u_1^m$

Now consider $M(D_3^m)$, each $u_1^i, i = 1, 2, 3 \ldots m$, will act as a root vertex. Color the vertices of $M(D_3^m)$ as follows.
Color the root vertex with $c_1$.

The inner subdivision vertices are colored with the color $c_2, c_3, \ldots, c_{2m+1}$ (clockwise)

The outer subdivision vertices are colored with $c_{3m-1}, c_{3m}, c_{3m+1}$ (clockwise)

The remaining ‘$2m$’ actual vertices are colored suitably with colors $3m+(j-1)$ and $3m+j$ for $1 \leq j \leq m$. It is verified that coloring is harmonious and the minimum number of colors required for harmonious coloring is $3m + 3$.

$\therefore \chi_H(M(D_3^m)) = 3m + 3$

Fig.3: Dutch windmill graph $D_3^3$ and $D_3^4$

Fig.4: Middle graph of Dutch Windmill graph $-M(D_3^3)$

$\chi_H(D_3^3) = 3m + 3 = 12$
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4. Graph theory, Encyclopedia of Britannica.


12. Wikipedia, the Free Encyclopedia.