A Grill view of nearly compact spaces

Karthika.A and I.Arockiarani
Nirmala College for Women
Coimbatore

Abstract

In this paper, a new class of topological spaces, namely, the class of $G$-nearly compact spaces ($G$-NC spaces) has been introduced. This class contains the class of $G$-compact spaces and is contained in the class of almost $G$-Compact spaces. Some of the properties of $G$-NC spaces have been obtained.

Keywords and phrases: $G$-near compactness, almost $G$-compactness, $G$-cover.

2000 AMS Subject Classification: 54C20, 54C99.

1. Introduction

Choquet [4] introduced an attractive theory of Grills, a collection satisfies some condition in Topological space. The topology equipped with the grill collection is called Grill topology. This topic has an excellent potential for application in other branches of mathematics like compactifications, Proximity spaces, different types of extension problems etc. This subject was continued to study by general topologists Roy and Mukherjee [8], [9] in recent years. It was probably the two articles of M.K. Singal and Asha Mathur [12], [13] that initiated the study of nearly Compact spaces in general topology.

The purpose of this paper is to define $G$-nearly compact spaces and investigate their basic properties.

Preliminaries

Definition 1.1 [8]

A collection $\mathcal{G}$ of nonempty subsets of a set $X$ is called a grill if

1. $A \in \mathcal{G}$ and $A \subseteq B \subseteq X$ implies that $B \subseteq \mathcal{G}$, and

2. $A \cup B \in \mathcal{G}$ ($A, B \subseteq X$) implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.
Definition 1.2 [9]

Let $G$ be a grill on a topological space $(X, \tau)$. A cover $\{U_{\alpha} : \alpha \in \Lambda\}$ of $X$ is said to be a $G$-cover if there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $X \setminus \bigcup_{\alpha \in \Lambda_0} U_{\alpha} \notin G$.

Definition 1.3 [8]

A grill $G$ on a set $X$ is said to be a $\sigma$-grill if for any countable collection $\{A_i : i \in I\}$ of subsets of $X$, $\bigcup_{i=1}^{\infty} A_i \notin G$ whenever $A_i \notin G$ for each $i \in I$.

Definition 1.4 [12]

A subset of a space is said to be regularly open, it is the interior of some closed set or equivalently, if it is the interior of its own closure. A set is said to be regularly closed if it is the closure of some open set or equivalently, if it is the closure of its own interior.

Definition 1.5 [1]

A topological space with the grill $G$ is said to be almost $G$-Compact if for every open cover $\mathcal{V} = \{U_{\alpha} : \alpha \in \Lambda\}$ there exists a finite subcollection $\{U_i : i \in I\}$ such that $X \setminus \bigcup_{i=1}^{n} \text{cl}U_i \notin G$.

Definition 1.6 [13]

A space $X$ is said to be semi-regular if every point of the space has a regular open neighbourhood.

Definition 1.7 [13]

A space $X$ is said to be almost-regular if for each point $x \in X$ and each neighbourhood $U$ of $x$, there exists a neighbourhood $V$ of $x$ such that $V \subset \text{cl}V \subset \text{intcl}U$.

Definition 1.8 [12]

A mapping is said to be almost-continuous if the inverse image of every regular open set is open.
Definition 1.9[13]

A mapping is said to be almost-open if it sends each regularly open set in to a open set.

Definition 1.10[13]

A mapping $f: X \rightarrow Y$ is said to be strongly continuous if $f[clA] \subset f[A]$ for every subset of $A$ of $X$.

Definition 1.11[13]

A mapping $f: X \rightarrow Y$ is said to be quasi-Compact if the image under $f$ of every open inverse subset of $X$ is open in $Y$.

Definition 1.12[12]

A set $P$ is said to be $\delta$-closed if for each point $x \in P$, there exists an open set $G$ containing $x$ such that $(intcl G) \cap P = \emptyset$, or equivalently, for each point $x \in P$, there exists a regularly open set containing $x$ which has intersection with $P$. A set is $\delta$-open if and only iff its compliment is $\delta$-closed.

Definition 1.13[12]

A point $x$ is a $\delta$-adherent point of a filter-base(filter) $F$ if $x \cap \{\{F\} : F \in F\}$. A filter-base (filter) $F$ $\delta$-converges to a point $x$ in $(X, \tau)$ if the interior of every closed neighbourhood of the point $x$ contains an $F \in F$ or equivalently, every $\delta$-open set containing $x$ contains an $F \in F$.

2. Near Compactness in grill

Definition 2.1

Let $G$ be a grill on the topological space $(X, \tau)$. Then $X$ is said to be Nearly compact space if for every open cover $V = \{U_\alpha : \alpha \in \Lambda\}$ of $X$, there exists a finite subcollection such that $X \setminus \cup_{i=1}^n intclU_i \notin G$.

Note 2.2

The class of $G$-nearly compact spaces contains the class of $G$-compact spaces and is
contained in the class of almost $G$-Compact spaces.

**Remark 2.3**

Every $G$-Compact space is $G$-Nearly Compact, but the converse is not true and this is shown in the following example.

**Example 2.4**

Let $\tau$ denote the co-countable topology on an uncountable set $X$ and $G$ be a grill of all uncountable sets of $X$. But $(X, \tau, G)$ is not $G$-Compact but $G$-Nearly Compact, and hence almost $G$-Compact.

**Definition 2.5**

A collection $\eta = \{A_\alpha : \alpha \in \Lambda\}$ of sets has the finite intersection property with respect to grill or simply $G$.f.i.p provided that the intersection of any finite subcollection belongs to grill. That is $\bigcap_{i=1}^n A_i \in G$.

**Theorem 2.6**

In a grill topological space $(X, \tau, G)$, with a $\sigma$ grill $G$, the following are equivalent:

(a) $(X, \tau, G)$ is a $G$-NC space.

(b) Every basic open cover of $X$ admits a finite subfamily, such that the difference from $X$ to the union of interiors of closures of whose members does not belong to the grill $G$.

(c) Every cover of $X$ by regularly open sets is a $G$-cover.

(d) Every family of regularly closed sets having the grill finite intersection property ($G$.f.i.p.) has arbitrary intersection belongs to the grill.

(e) Every family $\nu$ of closed sets having the property that for any finite subfamily $\bigcap_{i=1}^n \text{cl } \text{int } F_i \in G$ has intersection belongs to grill. That is $\bigcap_{i=1}^n F_i \in G$ implies that $\bigcap_{a \in \Lambda} F_a \in G$.

**Proof**

(a) $\Rightarrow$ (b): Obvious.

(b) $\Rightarrow$ (c): Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be any regularly open cover of $X$ and let $B$ be a base for $\tau$. 

R S. Publication, rspublicationhouse@gmail.com  Page 24
For each \( \alpha \in I \), \( U_\alpha = \bigcup \alpha \in \Lambda \alpha \{ V_\lambda : \lambda \in I \} \) is a basic \( \alpha \) open cover of \( X \). By (b), \( V \) has a finite sub-family, such that \( X \setminus \bigcup_{i=1}^n \text{intcl} V_i \notin G \). For each \( V_i \in V \), there exists an \( i \in I \), such that \( V_i \subseteq U_{ai} \Rightarrow \text{intcl} V_i \subseteq \text{intcl} U_{ai} = U_{ai} \).

Since \( G \) is a \( \sigma \)-grill and also \( X \setminus \bigcup_{i=1}^n U_{ai} \subseteq X \setminus \bigcup_{i=1}^n \text{intcl} V_i \), the finite subfamily \( \{ U_{ai} : i \in I \} \) satisfies the property, \( X \setminus \bigcup_{i=1}^n U_{ai} \notin G \). Hence the result (c).

(c)\( \Rightarrow \) (d): Let \( F = \{ F_\alpha : \alpha \in \Lambda \} \) be a family of regularly closed sets having the grill finite intersection property (\( G \).f.i.p.). If possible, let \( \cap_{\alpha \in \Lambda} F_\alpha \notin G \). Then \( \{ X \setminus F_\alpha : \alpha \in \Lambda \} \) is a regularly open cover of \( X \) and has a finite sub-cover \( \{ X \setminus F_{ai} : i = 1, 2, 3...n \} \), such that \( X \setminus \bigcup_{i=1}^n (X \setminus F_{ai}) \notin G \), \( \Rightarrow \cap_{i=1}^n F_{ai} \notin G \) which is a contradiction to our \( G \).f.i.p. Hence the result.

(d)\( \Rightarrow \) (e): Let \( F \) be a family of closed sets having the given property. Then \( \{ \text{clint} F : F \in F \} \) is a family of regular closed sets having the \( G \).f.i.p. By (d), \( \cap \{ \text{clint} F : F \in F \} \in G \). Also \( F \) being closed, \( \text{clint} F \subseteq F \) and \( G \) is a grill and Hence \( \cap \{ F : F \in F \} \in G \).

(e)\( \Rightarrow \) (a): Let \( U = \{ U_\alpha : \alpha \in \Lambda \} \) be any open cover of \( X \). If possible, let there be no finite subfamily \( U = \{ U_{ai} : i = 1, 2, 3...n \} \) of \( U \) such that \( X \setminus \bigcup_{i=1}^n \text{intcl} U_{ai} \notin G \), so that \( \cap (X \setminus \text{intcl}(U_{ai})) \in G \) for any finite subset. \( \Rightarrow \cap_{i=1}^n \text{cl int} (X \setminus U_{ai}) \in G \). by (e) \( \cap_{\alpha \in \Lambda} \text{cl int} (X \setminus U_{\alpha}) \in G \).\( \Rightarrow \cap_{i=1}^n \text{cl int} (X \setminus U_{ai}) \in G \).\( \Rightarrow \cap_{i=1}^n \text{cl int} (X \setminus U_{ai}) \in G \).\( \Rightarrow \) that \( \{ U_\alpha : \alpha \in \Lambda \} \) is a cover. Hence \( X \) is \( G \)-Nearly Compact.

Theorem 2.7

Let \( (X, \tau, G) \) be a semi-regular space with the \( \sigma \)-grill \( G \), then it is \( G \)-NC if and only if it is \( G \)-Compact.

Proof

Let \( (X, \tau, G) \) be a semi-regular space with the \( \sigma \)-grill \( G \), and let \( U = \{ U_\alpha : \alpha \in \Lambda \} \) be any open cover of \( X \). For each \( x \in X \), there exists an \( \alpha_x \in \Lambda \) such that \( x \in U_{ax} \). Since \( X \) is semi-regular, there exists a regularly open set \( G_x \) such that \( x \in G_x \subseteq U_{ax} \). Then \( \{ G_x : x \in X \} \) is a regularly open cover of \( X \) and has therefore a finite subcover \( \{ G_{xi} : i = 1, 2, 3...n \} \).
such that $X \setminus \bigcup_{i=1}^{n} G_{xi} \notin G$. Since the grill $G$ is a $\sigma$-grill and $G_{x} \subset U_{ax}$ . Then $\{U_{xi} : i = 1, 2, 3, ... n\}$ is a finite subcollection satisfies $X \setminus \bigcup_{i=1}^{n} U_{xi} \notin G$. Hence $X$ is $G$-Compact. The converse part is obvious, because every $G$-Compact space is $G$-nearly compact.

**Theorem 2.8**

Let $(X, \tau, G)$ be an almost -regular space with the $\sigma$-grill $G$. Then it is an almost $G$-compact space if and only if it is $G$-nearly Compact.

**Proof**

Let $U = \{U_{a} : a \in \Lambda\}$ be a regularly open cover of an almost regular, almost $G$-Compact space $(X, \tau, G)$. For each $x \in X$, there exists an $U_{ax} \in \Lambda$ such that $x \in U_{ax}$. By almost-regularity, there exists an open set $V_{x}$ such that $x \in V_{x} \subset cl(V_{x}) \subset intcl(U_{ax}) = U_{ax}$.

$V = \{V_{x} : x \in X\}$ is an open cover of $X$ which is an almost $G$-Compact space. Therefore there exists a finite sub-family such that $X \setminus \bigcup_{i=1}^{n} cl(V_{xi}) \notin G$. Since $G$ is a $\sigma$-grill and $V_{x} \subset U_{ax}$. Implies that $X \setminus \bigcup_{i=1}^{n} V_{axi} \notin G$. The converse is Obvious.

**Lemma 2.9**

Every open cover of regularly closed subset of a $G$ -NC space admits of a finite subfamily, the difference from $X$ to the interiors of the closures of whose members’ does not belongs to the grill $G$.

**Proof**

Let $(X, \tau, G)$ be a grill topological space. $U = \{U_{a} : a \in \Lambda\}$ be any open cover of regularly closed subset $Y$ of an $G$ -NC space $X$. Then $V \cup (X \setminus Y)$ is an open cover of $X$. There exists a finite subfamily $V = \{V_{\lambda} : \lambda \in I\}$ such that $X \setminus \bigcup_{\lambda \in I} intcl\{V_{\lambda} \cup (X \setminus Y)\} \notin G$.

$Y$ being regular closed set, $intcl(X \setminus Y) = X \setminus Y$. If $V_{\lambda} \subseteq X \setminus Y$ for no $\lambda \in \Lambda$, then $V$ is the required finite subfamily of $U$, such that $Y \setminus \bigcup_{\lambda \in \Lambda} V_{\lambda} \notin G$. Hence the proof.
3 Continuous mappings in nearly compact Grill topological spaces

This section examine the notion of continuous mappings in $G$–NC spaces. Further we introduce the concept of relative grill with respect to the subspace $(A, \tau_A)$ of the space $(X, \tau, G)$ and study its properties.

**Theorem 3.1**

Let $f: (X, \tau_1, G_1) \to (Y, \tau_2, G_2)$ be a function from $X$, which is a $G_1$-nearly compact space on to $Y$. If $f$ is a almost homeomorphism (bijective, almost-continuous, almost-open mapping) with $G_2 \subset f(G_1)$ and $f(G_1)$ is a $\sigma$-grill. Then $Y$ is a $G_2$-NC space.

**Proof**

Here $f: (X, \tau_1, G_1) \to (Y, \tau_2, G_2)$ is a almost homeomorphism mapping of an $G_1$-NC space $X$ on to $Y$. Let $U = \{ U_a : \alpha \in \Lambda \}$ be any regular open cover of $Y$. Then $f$ being almost continuous, $U^* = \{ f^{-1}(U_a) : \alpha \in \Lambda \}$ is an open cover of the $G_1$-NC space $X$. Therefore there exists a finite subfamily, $\{ f^{-1}(U_\alpha) : i = 1, 2, 3, \ldots, n \}$ of $U^*$ such that $X \setminus \bigcup_{i=1}^{n} \text{int} \{ f^{-1}(U_\alpha) \} \notin G_1$. Now, applying $f$ we get,

$f( X \setminus \bigcup_{i=1}^{n} \text{int} \{ f^{-1}(U_\alpha) \}) \notin f(G_1)$ since $f$ is on to, one to one and $G_2 \subset f(G_1)$ we have $Y \setminus f \{ \bigcup_{i=1}^{n} \text{int} \{ (U_\alpha) \} \} \notin G_2$

That is $Y \setminus \bigcup_{i=1}^{n} f \{ \text{int} \{ f^{-1}(U_\alpha) \} \} \notin G_2$ Since $f$ is almost open,

$Y \setminus \bigcup_{i=1}^{n} \text{int} \{ f(\text{cl}(U_\alpha)) \} \notin G_2$ Since $f$ is almost-continuous,

$Y \setminus \bigcup_{i=1}^{n} \text{int} \{ \text{cl}(U_{\alpha_i}) \} \notin G_2$

Since $f$ is being on to, $Y \setminus \bigcup_{i=1}^{n} (U_{\alpha_i}) \notin G_2$, $U_\alpha$’s being regular open.

Thus $U$ has a $G_2$-cover. Hence $Y$ is $G_2$-NC.

**Theorem 3.2**

If $f: (X, \tau_1, G_1) \to (Y, \tau_2, G_2)$ is a bijective, almost-continuous mapping with $G_2 \subset f(G_1)$, where $X$ is $G_1$-Compact and $f(G_1)$ is a $\sigma$-grill, then $Y$ is $G_2$-NC.
Proof

Let $f : (X, \tau_1, G_1) \rightarrow (Y, \tau_2, G_2)$ be an bijective, almost-continuous mapping of an $G_1$-Compact space $X$ onto a space $Y$. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a regular open cover of $Y$. Then $U = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is an open cover of $X$. Since $X$ is a $G_1$-Compact space, there exists a finite sub-family $\{f^{-1}(U_\alpha) : i = 1, 2, 3, ..., n\}$ such that, $X \setminus \bigcup_{i=1}^{n} f((U_\alpha)) \notin G_1$

Then, $f(X \setminus \bigcup_{i=1}^{n} ((U_\alpha))) \notin f(G_1)$. Since $G_2 \subset f(G_1)$ and $f$ is bijective, $Y \setminus \bigcup_{i=1}^{n} (U_\alpha) \notin G_2$. Hence the result.

Lemma 3.3

A space $(X, \tau, G)$ is almost $G$-Compact if and only if every regular open cover, $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ has a finite subfamily $\mathcal{U}^*$ such that $X \setminus \bigcup_{i=1}^{n} (\text{cl}(U_\alpha)) \notin G$, where $G$ is a $\sigma$-grill.

Proof

Let $(X, \tau, G)$ be a grill topological space. If a space $X$ is almost $G$-Compact and every regular open cover is also an open cover, then the condition of the lemma is obvious. To prove the converse, let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be any open cover of $X$. $\mathcal{V} = \{\text{int}clU_\alpha : \alpha \in \Lambda\}$ is then regular open cover, so that there exists a finite sub-family of $\mathcal{U}^*$ such that, there exists a corresponding sub-family of $\mathcal{U}^*$ such that $X \setminus \bigcup_{i=1}^{n} \text{cl}((U_\alpha)) \notin G$.

Then $X \setminus \bigcup_{i=1}^{n} \text{cl}((U_\alpha)) \notin G$, since $G$ is a $\sigma$-grill. Hence $X$ is almost $G$-Compact.

Theorem 4.2.4

Let $f : (X, \tau_1, G_1) \rightarrow (Y, \tau_2, G_2)$ be an bijective, almost-continuous mapping of an almost $G_1$-compact space with $G_2 \subset f(G_1)$ where $f(G_1)$ is a $\sigma$-grill. Then image of almost-continuous mapping is almost $G_2$-compact.

Proof

Let $f : (X, \tau_1, G_1) \rightarrow (Y, \tau_2, G_2)$ be an bijective, almost-continuous mapping of an almost $G_1$-compact space $X$ on to $Y$ and let $\{U_\alpha : \alpha \in \Lambda\}$ be any regular open cover of $Y$. 
\{f^{-1}(U_{\alpha}) : \alpha \in \lambda\} \text{ is then an open cover of } X. \text{ Since } X \text{ is almost } G_1\text{-Compact, therefore it has a finite sub-family } \{f^{-1}(U_{\alpha}) : i = 1, 2, 3, \ldots, n\} \text{ such that } X \setminus \bigcup_{i=1}^{n} (\text{cl } f^{-1}(U_{\alpha})) \notin G_1, f \text{ being almost continuous } \text{cl}[f^{-1}(U_{\alpha})] \subset f^{-1}\text{cl}(U_{\alpha}). \text{ Also } G_2 \subset f(G_1), G_1 \text{ is a } \sigma\text{-grill and } f \text{ is bijective, hence } Y \setminus \bigcup_{i=1}^{n} (\text{cl}(U_{\alpha})) \notin G_2. \text{ Thus } Y \text{ is almost } G_2\text{-Compact.}

**Corollary 3.5**

Let \( f : (X, \tau_1, G_1) \rightarrow (Y, \tau_2, G_2) \) be an bijective, almost-continuous mapping with \( G_2 \subset f(G_1) \) where \( G_1 \) is a \( \sigma\text{-grill}. \) Then,

1. If \( X \) is \( G_1\text{-Compact} \) then \( Y \) is almost \( G_2\text{-Compact}. \)
2. If \( X \) is \( G_1\text{-nearly compact} \) then \( Y \) is almost \( G_2\text{-Compact}. \)

**Theorem 3.6**

Let \( f : (X, \tau_1, G_1) \rightarrow (Y, \tau_2, G_2) \) be an bijective, strongly continuous mapping with \( G_2 \subset f(G_1) \) where \( f(G_1) \) is a \( \sigma\text{-grill}. \) Then the image of an almost \( G_1\text{-Compact} \) space is \( G_2\text{-Compact}. \)

**Proof**

Let \( f : (X, \tau_1, G_1) \rightarrow (Y, \tau_2, G_2) \) be an bijective, strongly continuous mapping with \( G_2 \subset f(G_1) \) where \( f(G_1) \) is a \( \sigma\text{-grill}. \) Let \( U = \{U_{\alpha} : \alpha \in \Lambda\} \) be any open cover of \( Y. \)

Then \( U = \{f^{-1}(U_{\alpha}) : \alpha \in \lambda\} \) is an open cover of \( X. \) Since \( X \) is almost \( G_1\text{-Compact} \) there exists a finite subfamily \( \{f^{-1}(U_{\alpha}) : i = 1, 2, 3, \ldots, n\} \) of \( U \) such that

\[
X \setminus \bigcup_{i=1}^{n} \text{cl}(f^{-1}(U_{\alpha})) \notin G_1
\]

That is as \( f \) is strongly continuous and \( f(G_1) \) is a \( \sigma\text{-grill}, \)

\[
X \setminus \bigcup_{i=1}^{n} f^{-1}(U_{\alpha}) \notin G_1. \text{ Then } Y \setminus \bigcup_{i=1}^{n} (U_{\alpha}) \notin G_2. \text{ Hence } \{U_{\alpha} : i = 1, 2, 3\ldots\} \text{ is a } G_2\text{-cover and so } Y \text{ is } G_2\text{-Compact.}
\]

**Corollary 3.7**

The property of being a \( G \text{-NC space} \) is preserved under bijective, strongly continuous mappings with the necessary conditions on the grill.
Theorem 3.8

An open subset $A$ of a space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-nearly compact if and only if every open cover of the set has a finite subfamily the interiors of the closures of whose members does not belong to the grill $\mathcal{G}$ where $\mathcal{G}$ is a $\sigma$-grill.

Proof

Let $A$ be an open $\mathcal{G}$-NC subset of $X$ and let $\{U_\alpha : \alpha \in \Lambda\}$ be any open cover of $X$. Then $\mathcal{U} = \{A \cap U_\alpha : \alpha \in \Lambda\}$ is then a relatively open cover of $A$. Let $\text{int}_A$ and $\text{cl}_A$ denote respectively the interior and closure operator in $A$. Since $A$ is $\mathcal{G}$-NC, there exists a finite subfamily $\{A \cap U_{\alpha i} : i = 1, 2, 3, \ldots, n\}$ of $\mathcal{U}$, such that

$$A \setminus \bigcup_{i=1}^{n} \text{int}_A \text{cl}_A (A \cap U_{\alpha i}) \notin \mathcal{G},$$

then $A \setminus \bigcup_{i=1}^{n} [A \cap \text{int}(A \cap U_{\alpha i})] \notin \mathcal{G}$. 

As $A$ is open, $\bigcup_{i=1}^{n} [A \cap \text{int}(A \cap U_{\alpha i})] \subset \bigcup_{i=1}^{n} \text{int}(U_{\alpha i})$. 

Since $\mathcal{G}$ is $\sigma$-grill, $A \setminus \bigcup_{i=1}^{n} \text{int}(U_{\alpha i}) \notin \mathcal{G}$. 

Conversely, any relatively open cover $\{V_\beta : \beta \in \Lambda\}$ of $A$ is open in the space also, as $A$ is open and hence has a finite subfamily $\{V_{\beta i} : i = 1, 2, 3, \ldots, n\}$ such that

$$A \setminus \bigcup_{i=1}^{n} \text{int}_A \text{cl}_A (V_{\beta i}) \notin \mathcal{G},$$

That is $A \setminus \bigcup_{i=1}^{n} [A \cap \text{int}(X_{\beta i})] \notin \mathcal{G}$. 

Imply that $A \setminus \bigcup_{i=1}^{n} [A \cap \text{int}(V_{\beta i})] \notin \mathcal{G}$. 

Then $A \setminus \bigcup_{i=1}^{n} \text{int}_A \text{cl}_A (V_{\beta i}) \notin \mathcal{G}$. 

Hence the result.

Theorem 3.9

Every clopen subset of a $\mathcal{G}$-nearly compact space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-nearly compact where $\mathcal{G}$ is a $\sigma$-grill.

Proof

Let $Y$ be any clopen subset of a $\mathcal{G}$-NC space $(X, \tau, \mathcal{G})$. Let $U$ be any open cover of $Y$ so that $U \cup (X \setminus Y)$ is an open cover of the $\mathcal{G}$-NC space $X$. There exists a finite subfamily $\mathcal{U}$ such that $X \setminus \bigcup_{i=1}^{n} \text{int}[U_{\alpha i} \cup (X \setminus Y)] \notin \mathcal{G}$. But $\text{int}(X \setminus Y) = (X \setminus Y)$.
Then \( X \setminus \bigcup_{i=1}^{n} \text{intcl}(U_{\alpha i}) \cup (X \setminus Y) \notin \mathcal{G} \). That is \( Y \setminus \bigcup_{i=1}^{n} \text{intcl}(U_{\alpha i}) \notin \mathcal{G} \) and hence \( Y \) is \(-\mathcal{G}\)-NC.

**Theorem 3.10**

The space \((X, \tau)\) with the grill \(\mathcal{G}\) is \(\mathcal{G}\)-nearly compact if and only if \((X, \tau^*)\) is \(\mathcal{G}\)-compact.

**Proof**

Let \((X, \tau, \mathcal{G})\) be a \(\mathcal{G}\)-nearly compact grill topological space. Let \(C\) be a basic \(\tau^*\)-open covering of \(X\). Then \(C\) is also a \(\tau\)-covering and hence there exists a finite subfamily, say \(\{C_i : i = 1, 2, 3, \ldots, n\}\) such that, \(X \setminus \bigcup_{i=1}^{n} \text{intcl}(C_i) \notin \mathcal{G}\). Since \(\text{intcl}C_i = C_i\), \(C\) has a finite subcovering and hence \((X, \tau^*)\) is \(\mathcal{G}\)-Compact. Conversely, let \(C\) be a \(\tau\)-covering. Then \(C \subseteq \text{intcl}C\) for all \(c \in C\), and therefore \(\{\text{intcl}C : c \in C\}\) is a \(\tau^*\)-covering and hence it has a finite subcollection (here \((X, \tau^*)\) is \(\mathcal{G}\)-Compact) such that \(X \setminus \bigcup_{i=1}^{n} \text{intcl}(C_i) \notin \mathcal{G}\). So, \((X, \tau)\) is \(\mathcal{G}\)-nearly Compact.

**Corollary 3.11**

For a grill topological space \((X, \tau, \mathcal{G})\), the following are equivalent:

(a) \(X\) is \(\mathcal{G}\)-nearly compact.

(b) Every \(\delta\)-open cover is a \(\mathcal{G}\)-cover.

(c) Every family of \(\delta\)-closed sets having finite intersection property with respect to the grill \(\mathcal{G}\) \((\mathcal{G}\text{-f.i.p})\) has the intersection belongs to the grill \(\mathcal{G}\).

(d) Every filter (filter-base) has a \(\delta\)-adherent point.

**Theorem 3.12**

A space is \(\mathcal{G}\)-nearly compact if and only if it is totally co- \(\mathcal{G}\)-Compact.

**Proof**

Let \((X, \tau)\) be a \(\mathcal{G}\)-nearly compact grill topological space. Let \(\tau_{\mathcal{G}}\) be any co-topology with the same grill collection \(\mathcal{G}\) of \(\tau\). Let \(C\) be a \(\tau_{\mathcal{G}}\)-open cover of \(X\).
Since \( B \subseteq \tau \), each \( X \setminus \text{cl} B \) is a regular open subset of \( X \). Therefore \( C \) is a \( \delta \)-open cover of \( X \) and thus there exists a finite sub-collection such that \( X \setminus \bigcup_{i=1}^{n} \text{int}(C_i) \notin \mathcal{G} \Rightarrow \) Since all \( C_i \) are regular open \( X \setminus \bigcup_{i=1}^{n} C_i \notin \mathcal{G} \). ⇒ since \( B \) is arbitrary, the co-space is a \( \mathcal{G} \)-Compact space for every \( B \).

Conversely, let \((X, \tau, \mathcal{G})\) be a totally co- \( \mathcal{G} \)-Compact space. Consider the co-topology \( \tau \) of \( \tau \). Then the space \((X, \tau)\) is \( \mathcal{G} \)-Compact. Let \( C \) be a regular-open cover of \((X, \tau)\). Then for each \( C \in \mathcal{C} \), \( C = \text{int}_\tau \text{cl}_\tau C = X \setminus (X \setminus \text{int}_\tau \text{cl}_\tau C) = X \setminus \text{cl}_\tau (X \setminus \text{cl}_\tau C) \). Since \( X \setminus \text{cl}_\tau C \in \tau \), \( C \in \tau \) and hence \( C \) is a \( \tau \)-open cover of \( X \) and thus it is a \( \mathcal{G} \)-cover ⇒ \((X, \tau, \mathcal{G})\) is \( \mathcal{G} \)-nearly Compact. Hence the result.

**Theorem 3.13**

A grill topological space is \( \mathcal{G}_2 \)-nearly Compact if and only if it is bijective, almost continuous image of a \( \mathcal{G}_1 \)-Compact space, with \( \mathcal{G}_2 \subset f(\mathcal{G}_1) \) and \( f(\mathcal{G}_1) \) is a \( \sigma \)-grill.

**Proof**

The bijective, almost-continuous image of a \( \mathcal{G}_1 \)-Compact space is \( \mathcal{G}_2 \)-NC with all the necessary conditions for grill (with \( \mathcal{G}_2 \subset f(\mathcal{G}_1) \) and \( f(\mathcal{G}_1) \) is a \( \sigma \)-grill).

Conversely, let \((X, \tau, \mathcal{G})\) be a \( \mathcal{G} \)-nearly compact space. Now \((X, \tau^*, \mathcal{G})\) is \( \mathcal{G} \)-Compact in the view of Theorem 5.4. Consider the identity mapping

\[
i: (X, \tau, \mathcal{G}) \rightarrow (X, \tau^*, \mathcal{G})\]

The mapping \( i \) is clearly almost continuous as every \( \tau \)-regular open set is \( \tau^* \)-open. Thus, there exists a \( \mathcal{G} \)-Compact space and an almost continuous mapping such that \((X, \tau, \mathcal{G})\) is the bijective, almost continuous image of that \( \mathcal{G} \)-Compact space under the condition \( \mathcal{G}_2 \subset f(\mathcal{G}) \). Hence the result.

**Definition 3.15**

Let \((X, \tau, \mathcal{G})\) be a grill topological space. Let \( A \) be a subset of \( X \). Then \( \mathcal{G}_A \) is called the relative grill of \( A \) with respective to the grill \( \mathcal{G} \) and \( \mathcal{G}_A = \{A \cap G : G \in \mathcal{G}, A \cap G \neq \emptyset\}\).

**Remark 3.16**

(1)In general two topologies \( \tau^*_{\mathcal{A}} \) and \((\tau _{\mathcal{A}})^*\) are not equivalent for any subset \( A \) of a grill
topological space \((X, \tau, G)\).

(2) A subset \(A\) is called \(\alpha\)-\(G\)-nearly compact if \((A, \tau_{\forall A}, G_A)\) is \(G_A\)-Compact.

(3) A subset \(A\) is called \(G\)-\(A\)-nearly compact if \((A, (\tau_{\forall A})^*, G_A)\) is \(G_A\)-Compact.

(4) In general \(G_A \not\subseteq G\). So, \(G_A\)-compact and \(G\)-compact are independent.

**Theorem 3.17**

An open subspace of a grill topological space is nearly compact if and only if it is nearly compact with respect to the relative grill of the subspace.

**Proof**

Let \((A, \tau_{\forall A}, G_A)\) be a subspace of a space \((X, \tau, G)\), which is either open or dense in \((X, \tau, G)\). Then for any subset \(B\) of \(A\), \(\text{int}_{\forall A} \text{cl}_{A} B = (\text{int} \text{cl}_{A} B) \cap A\). \(\Rightarrow \tau_{\forall A} = (\tau_{\forall A})^*\) and hence the result.

**Theorem 3.18**

For regular closed subsets, \(G_A\)-\(\alpha\)-near compactness implies \(G_A\)-near compactness.

**Proof**

Let \(A\) be a \(G_A\)-\(\alpha\)-nearly compact regular closed set of the grill topological space \((X, \tau, G)\). Let \(\tau\) be the family of \((\tau_{\forall A})^*\)-closed subsets of \(A\) with the finite intersection property with respect to the grill \((G.f.i.p)\). Since a \((\tau_{\forall A})\)-regular closed subset of a \(\tau\)-regular closed set \(A\) is \(\tau\)-regular closed, \(\tau\) is a family of \((\tau_{\forall A})^*\)-closed subsets of \(A\) with \(G.f.i.p\). Since \(A\) is a \(\alpha\)-\(G\)-nearly compact, \((A, \tau_{\forall A}, G_A)\) is \(G\)-compact and thus the arbitrary intersection belongs to the grill \(G_A\). Hence the result.

**Theorem 3.19**

If every subspace of a space \((X, \tau, G)\) is almost compact with respect to the relative grill, then \(X\) is semi-irreducible.

**Proof**

Let every open subspace is almost compact with respect to the relative grill and \(X\) is
not semi-irreducible. Then there is an infinite family \( \{ G_\alpha : \alpha \in \Lambda, G_\alpha \in \mathcal{G} \} \) of disjoint open sets.

Let \( G = \bigcup \{ G_\alpha : \alpha \in \Lambda \} \). Then \( G \) is almost \( \mathcal{G}_G \)-Compact subset, for the open cover \( \{ G_\alpha : \alpha \in \Lambda \} \) has no finite sub-family, such that \( X \setminus \bigcup_{\alpha \in \Lambda} \text{cl} G_\alpha \notin \mathcal{G}_G \). This is a contradiction for the fact that \( G \) is almost \( \mathcal{G}_G \)-compact. Therefore our claim is true. \( X \) is semi-irreducible.

**Theorem 3.20**

Let \((X, \tau, \mathcal{G})\) be a \( \mathcal{G} \)-nearly compact space. Let \( A \) be a closed set such that the boundary \( \partial A \) of \( A \) is \( \alpha \)-\( \mathcal{G} \)-nearly compact, then \( A \) is also \( \alpha \)-\( \mathcal{G} \)-nearly compact.

**Proof**

Let \( \{ C_\alpha : \alpha \in \Lambda \} \) be a \( \tau^* \)-\( \Lambda \)-open covering of \( A \). For each \( \alpha \in \Lambda \), there exists a \( G_\alpha \in \tau^* \) such that \( C_\alpha = A \cap G_\alpha \). For each \( x \in A \), there exists a \( C_{\alpha x} \) such that \( x \in C_{\alpha x} \) and hence a \( G_{\alpha x} \) such that \( x \in G_{\alpha x} \). Since \( G_{\alpha x} \in \tau^* \), there exists a \( \tau^* \)-regular open set \( V_{\alpha x} \), such that \( x \in V_{\alpha x} \subseteq G_{\alpha x} \). Now \( \{ V_{\alpha x} : x \in A \} \cup \{ X \setminus A \} \) forms a \( \tau^* \)-open cover of the \( \mathcal{G} \)-nearly compact space \( X \) and hence there exists a finite subfamily \( \{ V_{\alpha x} : i = 1, 2, 3, \ldots, n \} \) of \( \{ V_{\alpha x} : \alpha \in \Lambda \} \) such that,

\[
X \setminus \bigcup_{i=1}^{n} (\text{int cl} V_{\alpha xi}) \cup (\text{int cl}(X \setminus A)) \notin \mathcal{G}.
\]

Since \( \text{int cl}(X \setminus A) = X \setminus \text{cl int}(A) \) and \( \text{int} A \subseteq \text{cl} \text{int} A \), therefore \( \text{int} A \subseteq \bigcup \{ \text{int cl} V_{\alpha xi} : i = 1, 2, 3, \ldots, n \} = \bigcup \{ V_{\alpha xi} : i = 1, 2, 3, \ldots, n \} \subseteq \{ G_{\alpha xi} : i = 1, 2, 3, \ldots, n \} \).

Thus \( \text{int} A \subseteq \bigcup \{ G_{\alpha xi} \cap A : i = 1, 2, 3, \ldots, n \} = \bigcup \{ C_{\alpha xi} : i = 1, 2, 3, \ldots, n \} \). Now \( \{ G_\alpha \cap \partial A : \alpha \in \Lambda \} \) is a \( \tau^* \)-\( \partial A \)-open covering of the \( \alpha \)-\( \mathcal{G} \) nearly \( G_{\alpha x} \)-compact set \( \partial A \) and hence there exists a finite subfamily \( \{ G_{\alpha xj} : j = 1, 2, 3, \ldots, m \} \) of \( \{ G_\alpha \cap \partial A : \alpha \in \Lambda \} \) such that,

\[
\partial A \setminus \bigcup_{j=1}^{m} \{ (G_{\alpha xj}) \cap \partial A \} \notin \mathcal{G}.
\]

Now \( A = (A \setminus \text{int} A) \cup \text{int} A = \partial A \cup \text{int} A, (\text{as } A \text{ is closed}) \Rightarrow \\
A \setminus \{ \bigcup \{ C_{\alpha xj} : j = 1, 2, 3, \ldots, m \} \} \cup \{ \bigcup \{ C_{\alpha xi} : i = 1, 2, 3, \ldots, n \} \} \notin \mathcal{G}.
\]

Hence \( A \) is \( \alpha \)-\( \mathcal{G} \)-nearly compact.
References


